

An intermediate value theorem for \mathbb{Q}

The intermediate value theorem does not hold for continuous functions on closed intervals of rational numbers, not even if we restrict to rational intermediate values. Take for example the function x^2 on $\mathbb{Q} \cap [2, 3]$: this function is continuous and there is the rational intermediate value $5 \in [4, 9]$ however there is no rational number whose square equals 5.

One problem is that \mathbb{Q} is not complete: there are Cauchy sequences of rational numbers that do not converge to a rational number. Another problem is that a closed interval of rational numbers is not compact (unless it is a point) and hence a continuous function with that domain may fail to be uniformly continuous.

Theorem. Consider a uniformly continuous real-valued function f with domain $D = \mathbb{Q} \cap [a, b]$, where $a < b$ are rational numbers. For every $r \in [\min(f(a), f(b)), \max(f(a), f(b))]$ there is a Cauchy sequence x_n of elements of D such that $f(x_n)$ converges to r .

Proof. Up to working with $-f$ instead, we may suppose $f(a) \leq f(b)$. By induction we build two Cauchy sequences a_n and b_n of rational numbers in D with the same limit and with the property that $f(a_n) \leq r \leq f(b_n)$ holds for all $n \in \mathbb{N}$. Set $a_0 := a$ and $b_0 := b$. For the inductive step, comparing r and $f(\frac{1}{2}(a_n + b_n))$, we set (a_{n+1}, b_{n+1}) to be either $(a_n, \frac{1}{2}(a_n + b_n))$ or $(\frac{1}{2}(a_n + b_n), b_n)$ so that the requested property holds for $n + 1$. The sequence $a_n - b_n$ tends to zero, and hence the same holds for the sequence $f(a_n) - f(b_n)$ because f is uniformly continuous. Since $f(a_n) \leq r \leq f(b_n)$, both sequences $f(a_n)$ and $f(b_n)$ converge to r . \square

An intermediate value theorem for \mathbb{Z}

The intermediate value theorem does not hold for sequences, even if we restrict to integer values. Assuming uniform continuity we can make sure that the sequence makes only small jumps, i.e. varies gradually/stepwise:

Theorem. Consider an integer-valued function f with domain $D = \mathbb{Z} \cap [a, b]$, where $a < b$ are integers. Suppose that for every $n \in \mathbb{Z}$ such that $a \leq n < b$ we have

$$|f(n+1) - f(n)| \leq 1.$$

Then for every integer $z \in [\min(f(a), f(b)), \max(f(a), f(b))]$ there is some $x \in D$ such that $f(x) = z$.

Proof. This is a consequence of the next result because $|f(x) - z| \leq \frac{1}{2}$ implies $f(x) = z$. \square

Theorem. Consider a real-valued function f with domain $D = \mathbb{Z} \cap [a, b]$, where $a < b$ are integers. Suppose that for every $n \in \mathbb{Z}$ such that $a \leq n < b$ we have

$$|f(n+1) - f(n)| \leq c$$

for some positive real constant c . Then for every $r \in [\min(f(a), f(b)), \max(f(a), f(b))]$ there is some $x \in D$ such that $|f(x) - r| \leq \frac{c}{2}$.

Proof. Suppose (up to working with $-f$ instead) that $f(a) \leq f(b)$. Consider the smallest integer $z \in [a, b]$ satisfying $f(z) \geq r$. We have $z = a$ only if $f(a) = r$ and in this case the statement holds. If $z > a$ then $z - 1$ is also in the domain and we have $f(z - 1) < r \leq f(z)$. We may then take $x = z$ or $x = z - 1$ because by assumption $|f(z) - f(z - 1)| \leq c$. \square