Arithmetic billiards in dimension n

We assume that is the reader is familiar with the setting of the two-dimensional arithmetic billiards.

Theorem 1. *Let us call x_1, x_2, ..., x_n, the (positive and integer) variables, and let a_1, a_2, ..., a_n, their maximal value. Then the total length of the path in a $n-$dimensional cuboid is equal to* $\operatorname{lcm}(a_1,\, a_2, ...,\, a_n)$.

Proof. First of all, when the path hits one corner, the coordinates of that corner must be:

$$\begin{cases} x_1 = 0 \;\; \textit{or} \;\; x_1 = a_1 \\ x_2 = 0 \;\; \textit{or} \;\; x_2 = a_2 \\ . \\ . \\ . \\ x_n = 0 \;\; \textit{or} \;\; x_n = a_n \end{cases}$$

Let us now call c_1, c_2, ..., c_n, the coordinates of the hit corner. Then we know that c_1, c_2, ..., c_n have to be a positive common multiple of a_1, a_2, ..., a_n, so at least $\operatorname{lcm}(a_1,\, a_2, ...,\, a_n)$. On the other hand, after that amount of unitary steps in the path (independently of the reflections) each coordinate x_i is a multiple of a_i and hence we are in a corner. □

Let $I = \{1, \ldots, n\}$ and for every non-empty subset $J \subseteq I$ write $\operatorname{lcm}(a_J)$ for the least common multiple of the numbers a_i with $i \in J$.

Theorem 2. *Let us call x_1, x_2, ..., x_n, the (positive and integer) variables, and let a_1, a_2, ..., a_n, their maximal value. The total amount of bouncing points in the $n-$dimensional arithmetic billiard is:*

$$\operatorname{lcm}(a_I) \cdot \sum_{J \subseteq I, J \neq \emptyset} (-1)^{\#J+1} \frac{1}{\operatorname{lcm}(a_J)}$$

Proof. The number of bouncing points on the faces $x_i = 0$ or $x_i = a_i$ equals $\frac{\operatorname{lcm}(a_1, a_2, \ldots, a_n)}{a_i}$. Similarly, on an edge $x_i = 0$ or $x_i = a_i$ for all $i \in J$, where J is a non-empty subset of $\{1, \ldots, n\}$, equals $\frac{\operatorname{lcm}(a_I)}{\operatorname{lcm}(a_J)}$. This formula is then obtained by the inclusion-exclusion principle while counting the bouncing points on the faces (we have to keep track of points which are bouncing points for several faces simultaneously). □

Open questions

For an n-th dimensional billiard with the path starting at the point $(0, \ldots, 0)$ and each coordinate increasing or decreasing by one at each step:

- In which corner does the ball land?
  This question is probably clarified by considering the highest power of 2 dividing the numbers a_i.
- How many intersection points does the path have?
  Many examples need to be investigate to formulate a conjecture.

In general, what happens if the starting point is not $(0, \ldots, 0)$ but another point in the arithmetic billiard?