VISUALIZATIONS FOR THE PRINCIPLE OF MATHEMATICAL INDUCTION

The Principle of Mathematical Induction (in short, PMI) is the mathematical formalization of a "chain reaction": one event sets off a chain of events. The first event is called *base case*, and what guarantees the propagation of the reaction is called *induction step*. The PMI is usually illustrated by a row of falling dominoes [4]:

There is a "classical" Principle of Mathematical Induction, and several variants: we investigate these variants, and provide intuitive visualizations. The classical PMI can be considered to be either an axiom or a theorem, being equivalent to the fact that every non-empty subset of the natural numbers has a least element. All the variants can be proven with the classical PMI.

Mathematical induction is a very convenient method of proof, which can be applied to all areas of mathematics. It is a topic of most calculus courses, and it is also helpful for mathematical competitions. Comprehensive texts are [1, 3], online references are for example [2, 5].

THE CLASSICAL MATHEMATICAL INDUCTION

Consider a mathematical chain reaction, where the events constituting the reaction are statements that are proven to be true, i.e. properties that turn out to hold. More precisely, we have a collection of statements

$$P(n) \qquad \text{with } n \in \mathbb{N}$$

and the aim is proving that all statements hold true. Notice that there is a first statement $P(0)$, for every statement there exists a unique successor ($P(n) \rightsquigarrow P(n+1)$), and each statement is eventually reached in this way starting from the first one.
The Principle of Mathematical Induction (in short, PMI) is the guarantee that a *base case* and an *induction step* are sufficient to ensure that all $P(n)$ hold true: the base case means proving the first statement $P(0)$, while the induction step means proving the following implication for all $n \in \mathbb{N}$:

$$\text{if } P(n) \text{ holds true, then } P(n+1) \text{ holds true}\,.$$

• **PMI Classic:** Consider statements $P(n)$ for $n \in \mathbb{N}$. Suppose that $P(0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ the statement $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

Exercise 1: Prove that for every $n \in \mathbb{N}$ the number $n^3 - n$ is a multiple of 3.

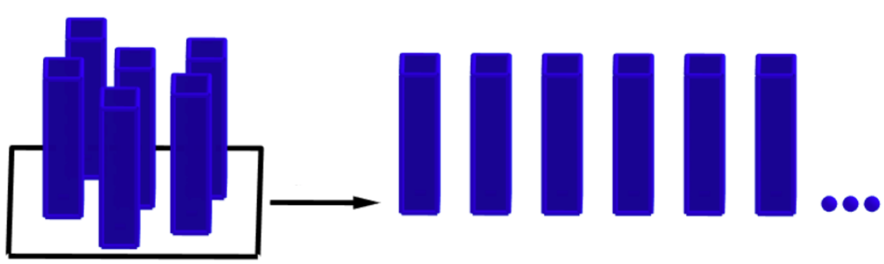
The typical visualization of PMI Classic consists in a row of falling dominoes:

Consider infinitely many (to be precise, countably many) dominoes standing on end, and arranged in a half-line extending infinitely to the right. Looking from left to right, the first domino represents $P(0)$ and the $(n+1)$th domino represents $P(n)$. Proving the truthfulness of $P(n)$ means that the corresponding domino falls to the right. The base case means pushing the first domino as to make it fall, and the induction step means that, if one domino falls, then its right-hand neighbor falls as well. Thus the base case starts a chain reaction of falling dominoes, while the induction step guarantees that the chain reaction includes all dominoes in the row: eventually each domino will fall.
Without the base case i.e. without a push the dominoes keep standing. A missing induction step can be visualized by a row that at some point is not tight: if there is too much space between a domino and the next one, then the chain reaction will not propagate.

MATHEMATICAL INDUCTION ON A COUNTABLE SET

• **PMI Countably Infinite:** If the set of statements is countably infinite, then it suffices to label its elements with the natural numbers to reduce to the situation of PMI Classic.

Visually, we are arranging the dominoes in a row:

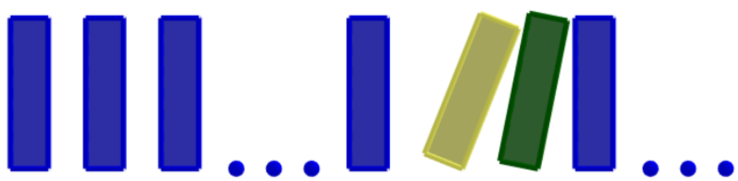


For the set of even non-negative integers we typically choose 0 as first element, 2 as second, 4 as third, and so on (for the odd non-negative integers we would choose 1 as first element, 3 as second, 5 as third, and so on). For the set of integers smaller than or equal to -5 it is natural to take -5 as as first element, -6 as second, -7 as third, and so on. If we have all integers, then we could order them as follows: $0, 1, -1, 2, -2, 3, -3 \ldots$

A special case of PMI Countably Infinite is the following:

- **PMI Different Start:** Let $n_0 \in \mathbb{N}$, and consider statements $P(n)$ for $n \in \mathbb{N}$ with $n \geqslant n_0$. Suppose that $P(n_0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ with $n \geqslant n_0$ the statement $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$ with $n \geqslant n_0$.

For PMI Different Start we have the whole row of dominoes indexed by the natural numbers (by considering some additional statements) and push the domino corresponding to n_0. In this way the first dominoes stay untouched, while all others will fall. We simply ignore the first dominoes: these could either fall if pushed (the statements are true) or they are fixed (the statements are false).



Exercise 2: Prove that for all natural numbers $n \geqslant 4$ we have $n \cdot (n-1) \cdots 2 \cdot 1 > 2^n$.

For a *non-empty finite set* of statements (for an empty set there would be nothing to prove) we have:

- **PMI Finite:** Let S be a non-empty finite set, and consider statements $P(s)$ for $s \in S$. We label the elements of S with the natural numbers from 0 to $c-1$, where c is the cardinality of S. Suppose that $P(0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ with $0 \leqslant n < c-1$ the statement $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(s)$ holds true for all $s \in S$.

Visually, the row of dominoes is finite: after finitely many steps all dominoes have fallen.

Exercise 3: Prove that for all integers n in the range from 20 to 50 the binomial coefficient $\binom{30}{n-20}$ is strictly positive.

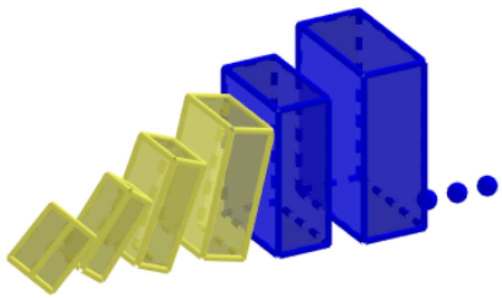
Further variants of the PMI can be combined with PMI Countably Infinite or PMI Finite.

THE COMPLETE MATHEMATICAL INDUCTION

• **PMI Complete:** Consider statements $P(n)$ for $n \in \mathbb{N}$. Suppose that $P(0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ the collection of statements $P(0)$ to $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

The induction step is easier to establish with respect to PMI Classic because we can make use of any statement from $P(0)$ to $P(n)$. In many situations we only need a fixed amount of previous statements, for example we may only need $P(n)$ and $P(n-1)$. Further variants of the PMI can be combined with PMI Complete.

We may visualize PMI Complete with a row of dominoes of growing size:



The induction step means that the first dominoes together have enough elain to make the next domino fall (one domino alone may not be sufficiently heavy to push down the next one).

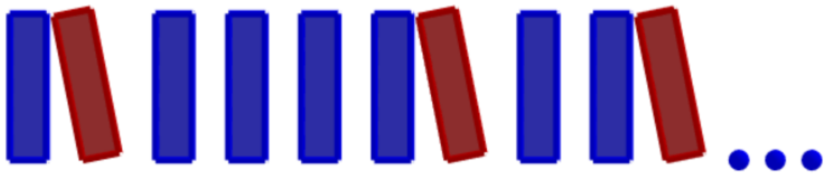
Exercise 4: The Fibonacci number sequence F_n can be defined by setting $F_0 = 0$ and $F_1 = 1$, and by requiring that $F_n = F_{n-2} + F_{n-1}$ holds for all $n \geqslant 2$. Prove that for all $n \in \mathbb{N}$ we have

$$F_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big].$$

THE BACKWARDS MATHEMATICAL INDUCTION

• **PMI Backwards:** Consider statements $P(n)$ for $n \in \mathbb{N}$. Suppose that $P(n)$ is true for all $n \in S$, where S is an infinite subset of $\mathbb{N}$ (this is an infinite set of base cases). Suppose that for every $n \in \mathbb{N}$ with $n > 0$ the statement $P(n)$ implies $P(n-1)$ (this is the backward induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

We are doing infinitely many applications of PMI Finite, one for each element of S (each statement is then proven multiple times). Visually, we are pushing to the left all dominoes corresponding to the elements of S: the chain reaction now propagates to the left.



Exercise 5: Let $n \in \mathbb{N}$ with $n \geqslant 1$. Prove the inequality between arithmetic and geometric mean of n strictly positive real numbers:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \ldots a_n}\,.$$

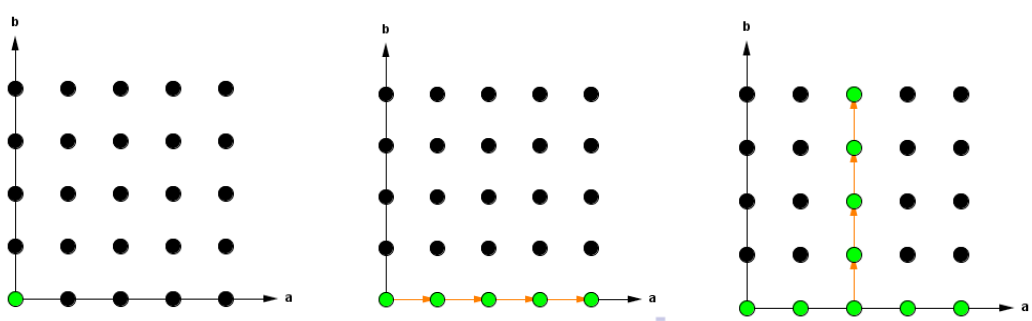
(Hint: Prove the inequality by induction for all n that are powers of 2.)

TWO-DIMENSIONAL INDUCTION

• **PMI Two-dimensional:** Consider statements $P(a,b)$ for $a, b \in \mathbb{N}$. Suppose that $P(0,0)$ is true (this is the base case). Suppose that, if $P(a,0)$ is true for some $a \in \mathbb{N}$, then $P(a+1,0)$ is also true (this is the first induction step). Suppose that, if $P(a,b)$ is true for some $a, b \in \mathbb{N}$, then $P(a, b+1)$ is also true (this is the second induction step). Then $P(a,b)$ holds true for every $a, b \in \mathbb{N}$.

We apply PMI Classic, once on the first variable and infinitely many times on the second variable: with the help of the first induction step one proves $P(a,0)$ for all $a \in \mathbb{N}$; the second induction step then allows to take $a \in \mathbb{N}$ and prove $P(a,b)$ for that fixed a and for any $b \in \mathbb{N}$.

As soon as $P(a,b)$ is proven, we mark the point (a,b) in the plane. We mark $(0,0)$ because of the base case and then, with the help of the first induction step, we mark all points on the a-axis. With the second induction step the marking propagates upwards from any point $(a,0)$, thus the marking propagates to all points (a,b) with $a, b \in \mathbb{N}$.



Exercise 6: Consider a function $f(a,b)$ of two strictly positive integer variables that satisfies $f(1,1) = 2$ and such that for every a, b the following holds:

$$f(a+1,b) = f(a,b) + 2(a+b) \qquad \text{and} \qquad f(a,b+1) = f(a,b) + 2(a+b-1)\,.$$

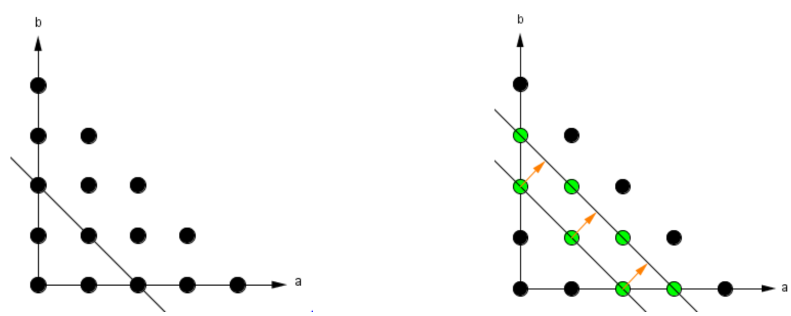
Prove that for every a, b we have $f(a,b) = (a+b)^2 - (a+b) - 2b + 2$.

We can easily generalize PMI Two-dimensional to finitely many variables.

Grouping statements

• **PMI Sum of variables:** Consider statements $P(a,b)$ for $a, b \in \mathbb{N}$. Suppose that $P(0,0)$ is true (this is the base case). Suppose that, if for some $n \in \mathbb{N}$ the statement $P(a,b)$ is true whenever $a+b=n$, then the statement $P(a,b)$ is true whenever $a+b=n+1$ (this is the induction step). Then $P(a,b)$ holds true for every $a, b \in \mathbb{N}$.

We apply PMI Classic to the statements $Q(n)$ consisting of all $P(a,b)$ with $a+b=n$. We are thus grouping statements together, namely those corresponding to the points on a same finite diagonal of the first quadrant in the plane. The chain reaction propagates from one diagonal to the next. The two pictures below represent $Q(2)$ (which contains the statements $P(2,0)$, $P(1,1)$, and $P(0,2)$) and the induction step from $Q(2)$ to $Q(3)$:

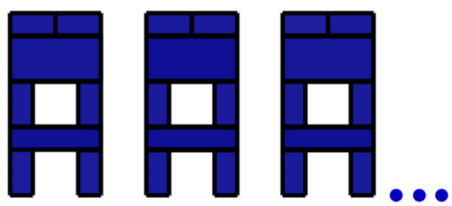


Exercise 7: Prove that for all natural numbers n, k such that $k \leqslant n$ the binomial coefficient $\binom{n}{k}$ is a natural number. You can make use of the known formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
(Hint: Apply PMI Complete.)

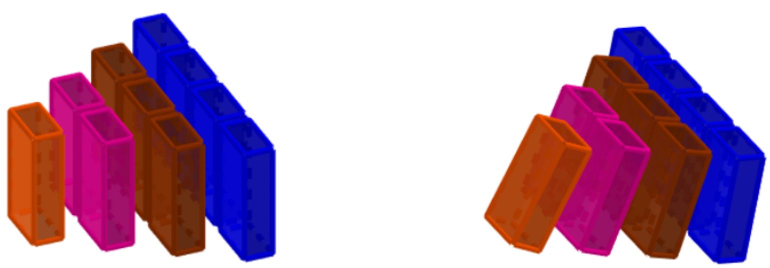
The general framework for grouping statements is as follows. Consider statements $P(s)$ with s varying in a set S, and partition S into countably many subsets (for example, partition the real numbers into countably many intervals). We leave to the reader the case of a finite partition, which requires PMI Finite. So let us denote by T_n for $n \in \mathbb{N}$ the subsets of the partition: we group the statements $P(s)$ for $s \in T_n$ and apply PMI Classic.

• **PMI Partition:** Let S be a set, and consider statements $P(s)$ for $s \in S$. Partition S into countably many subsets T_n with $n \in \mathbb{N}$. Suppose that $P(s)$ is true for all $s \in T_0$ (this is the base case). For all $n \in \mathbb{N}$ suppose that, if $P(s)$ is true whenever $s \in T_n$, then $P(s)$ is true whenever $s \in T_{n+1}$ (this is the induction step). Then $P(s)$ holds true for every $s \in S$.

Domino towers represent the grouping of statements: the towers completely fall apart in the chain reaction, i.e. all their dominoes fall (to apply PMI Complete consider towers of growing size).



More generally consider an arrangement of dominoes: if all dominoes in some arrangement fall, then all dominoes in the next arrangement fall.



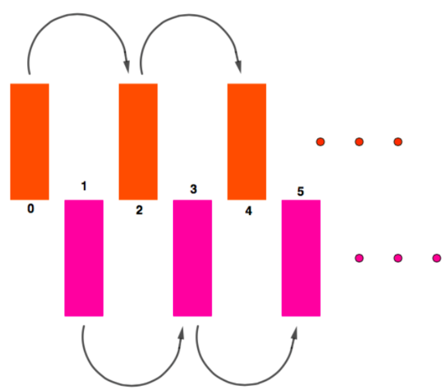
Exercise 8: For all finite subsets F of $\mathbb{N}$, prove that the number of subsets of F equals $2^{\#F}$.

The sets of a partition are pairwise disjoint, so we never repeat statements in PMI Partition. However, proving a statement multiple times is not wrong, and sometimes increasing the number of statements that one has to prove turns out to be practical. Also notice that sometimes it is possible to fix some of the objects appearing in a problem and apply the mathematical induction to the remaining ones.

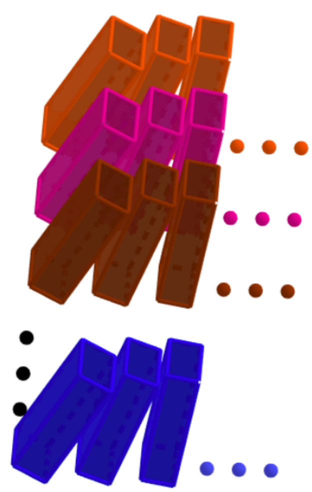
Larger induction step

• **PMI Jumps:** Consider statements $P(n)$ for $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ with $k \geqslant 1$. Suppose that the statements $P(0)$ up to $P(k-1)$ are true (we have k base cases). Suppose that for every $n \in \mathbb{N}$ the statement $P(n)$ implies $P(n+k)$ (in the induction step we jump k steps ahead). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

The set $\mathbb{N}$ is partitioned into k subsets, according to the remainder after division by k. We then apply k times PMI Countably infinite. In particular, for $k = 2$ there is one induction for the even numbers, and one for the odd numbers:



We may visualize PMI Jumps with k rows of falling dominoes: hitting the next domino in the row means jumping k steps ahead in the usual arrangement.



An alternative to PMI Jumps is doing a **case distinction in the proof of the induction step** of PMI Classic. For example, instead of applying PMI Jumps for $k = 2$, one could prove the induction step for the case "from even to odd" and for the case "from odd to even".

Exercise 9: Prove that for all $n \in \mathbb{N}$ we have

$$(-1)^n = \begin{cases} 1 & for\ n\ even\,; \\ -1 & for\ n\ odd\,. \end{cases}$$

Similarly, proving by induction the formula for the higher derivatives of the sinus function requires either PMI Jumps with $k = 4$ or a case distinction with four cases.

The visualization for the case distinction is that not all dominoes fall down in the same way: the row of dominoes goes in a zigzag, or the dominoes are not aligned (in the following picture the dominoes are seen from above).

Summary: The Principle of Mathematical Induction is usually illustrated with a row of falling dominoes. We investigate several variants of this principle, providing intuitive visualizations for them.

References

[1] T. Andreescu, V. Crişan. *Mathematical Induction: A Powerful and Elegant Method of Proof*, XYZ Press, 2017.

[2] S. Chakraborty, *Mathematical Induction*, Lecture 18 of CSE20 (Spring 2014), 15 pages, available for download at https://cseweb.ucsd.edu/classes/sp14/cse20-a/InductionNotes.pdf, accessed 1 August 2018.

[3] D. S. Gunderson, *Handbook of Mathematical Induction: Theory and Applications*, Chapman & Hall/CRC, 2010.

[4] Louise Docker Photography, Flickr Image (cropped) https://commons.wikimedia.org/wiki/File:Domino_effect.jpg#/media/File:Domino_effect.jpg, available under the CCBY 2.0 license, uploaded by Ranveig on 3 August 2007.

[5] Wikipedia contributors, *Mathematical Induction*, Wikipedia, The Free Encyclopedia, 16 June 2018, https://en.wikipedia.org/wiki/Mathematical_induction, accessed 1 August 2018.

SOLUTIONS TO THE EXERCISES FOR THE READER

• **Exercise 1:** Prove that for every $n \in \mathbb{N}$ the number $n^3 - n$ is a multiple of 3.

Solution (with PMI Classic): For $n = 0$ the number $n^3 - n$ is zero and hence a multiple of 3. If $n^3 - n$ is a multiple of 3, then the number $(n+1)^3 - (n+1) = (n^3 - n) + 3(n^2 + n)$ is also a multiple of 3.

• **Exercise 2:** Prove that for all natural numbers $n \geqslant 4$ we have $n \cdot (n-1) \cdots 2 \cdot 1 > 2^n$.

Solution (with PMI Different Start): For $n = 4$ we have $4 \cdot 3 \cdot 2 \cdot 1 = 24 > 16 = 2^4$ so the statement is true. If for some $n \geqslant 4$ we have $n \cdot (n-1) \cdots 2 \cdot 1 > 2^n$, then $(n+1) \cdot n \cdot (n-1) \cdots 2 \cdot 1 > (n+1) \cdot 2^n > 2^{n+1}$.

• **Exercise 3:** Prove that for all integers n in the range from 20 to 50 the binomial coefficient $\binom{30}{n-20}$ is strictly positive.

Solution (with PMI Finite): The base case $n = 20$ is clear because $\binom{30}{0} = 1 > 0$. To prove the induction step for $20 \leqslant n < 50$ suppose that the assertion holds for n, i.e. that $\binom{30}{n-20} > 0$. We can write

$$\binom{30}{n+1-20} = \binom{30}{n-20} \cdot \frac{30 - (n-20)}{n+1-20}$$

and hence the binomial coefficient on the left-hand side, being the product of two strictly positive integers, is also strictly positive. This proves the statement for $n + 1$.

• **Exercise 4:** The Fibonacci number sequence F_n can be defined by setting $F_0 = 0$ and $F_1 = 1$, and by requiring that $F_n = F_{n-2} + F_{n-1}$ holds for all $n \geqslant 2$. Prove that for all $n \in \mathbb{N}$ we have

$$F_n = \frac{1}{\sqrt{5}} \Big[\Big(\frac{1+\sqrt{5}}{2} \Big)^n - \Big(\frac{1-\sqrt{5}}{2} \Big)^n \Big] .$$

Solution (with PMI Complete): The assertion for $n = 0$ is evident. To prove the induction step for $n = 0$, we can show directly that the given expression for $n = 1$ outputs $F_1 = 1$. To prove the induction step for $n \geqslant 2$ we can make use of the recursion formula $F_{n+1} = F_{n-1} + F_n$ and of the statement for F_{n-1} and F_n:

$$\begin{aligned} F_{n+1} &= \frac{1}{\sqrt{5}} \Big[\Big(\frac{1+\sqrt{5}}{2} \Big)^{n-1} - \Big(\frac{1-\sqrt{5}}{2} \Big)^{n-1} \Big] + \frac{1}{\sqrt{5}} \Big[\Big(\frac{1+\sqrt{5}}{2} \Big)^{n} - \Big(\frac{1-\sqrt{5}}{2} \Big)^{n} \Big] \\ &= \frac{1}{\sqrt{5}} \Big[\Big(\frac{1+\sqrt{5}}{2} \Big)^{n-1} \Big(\frac{3+\sqrt{5}}{2} \Big) - \Big(\frac{1-\sqrt{5}}{2} \Big)^{n-1} \Big(\frac{3-\sqrt{5}}{2} \Big) \Big] \\ &= \frac{1}{\sqrt{5}} \Big[\Big(\frac{1+\sqrt{5}}{2} \Big)^{n+1} - \Big(\frac{1-\sqrt{5}}{2} \Big)^{n+1} \Big] . \end{aligned}$$

• **Exercise 5:** Let $n \in \mathbb{N}$ with $n \geqslant 1$. Prove the inequality between arithmetic and geometric mean of n strictly positive real numbers:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \ldots a_n} .$$

(Hint: Prove the inequality by induction for all n that are powers of 2.)

Solution (with PMI Backwards): We first prove the assertion for all $n \in S$, where S is the set of powers of 2. Consider as statement $P(n)$, for $n \in \mathbb{N}$ with $n \geqslant 1$, the given inequality for 2^n numbers. The statement $P(1)$ is easy to prove because we have the following equivalences

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \iff \Big(\frac{a_1 + a_2}{2} \Big)^2 \geq a_1 a_2 \iff (a_1 - a_2)^2 \geqslant 0$$

and the last inequality is trivially true. Now suppose that $P(n)$ is true, which means

$$\frac{a_1 + a_2 + ... + a_{2^n}}{2^n} \geqslant \sqrt[2^n]{a_1 a_2 ... a_{2^n}}\,.$$

We have to prove that $P(n+1)$ is true: by the known statement $P(1)$ and applying $P(n)$ we get

$$\begin{aligned} \frac{a_1 + a_2 + ... + a_{2^{n+1}}}{2^{n+1}} &= \frac{1}{2^n}\left(\frac{a_1+a_2}{2} + \frac{a_3+a_4}{2} + ... + \frac{a_{2^{n+1}-1} + a_{2^{n+1}}}{2}\right) \\ &\geqslant \frac{\sqrt{a_1a_2} + \sqrt{a_3a_4} + ... + \sqrt{a_{2^{n+1}-1}a_{2^{n+1}}}}{2^n} \\ &\geqslant \sqrt[2^n]{\sqrt{a_1a_2}\sqrt{a_3a_4}...\sqrt{a_{2^{n+1}-1}a_{2^{n+1}}}} \\ &= \sqrt[2^{n+1}]{a_1a_2...a_{2^{n+1}}}\,. \end{aligned}$$

Now we prove the induction step for PMI Backwards: supposing that the inequality holds for n numbers, we show that it holds for $n-1$ numbers. Consider strictly positive real numbers $a_1, \dots, a_{n-1}$, and set $a_n := \frac{a_1+a_2+\dots+a_{n-1}}{n-1}$ (notice that a_n is again a strictly positive real number). The inequality for n numbers then gives

$$\frac{a_1 + a_2 + ... + a_{n-1} + \frac{a_1+a_2+...+a_{n-1}}{n-1}}{n} \geqslant \sqrt[n]{a_1a_2...a_{n-1}\cdot\left(\frac{a_1+a_2+...+a_{n-1}}{n-1}\right)}.$$

Raising the above inequality to the n-th power yields

$$\left(\frac{a_1+a_2+...+a_{n-1}}{n-1}\right)^n \geqslant a_1a_2...a_{n-1}\cdot\left(\frac{a_1+a_2+...+a_{n-1}}{n-1}\right)$$

and hence

$$\left(\frac{a_1+a_2+...+a_{n-1}}{n-1}\right)^{n-1} \geqslant a_1a_2...a_{n-1}\,.$$

Taking the $(n-1)$th square root we obtain the requested inequality for $n-1$ numbers.

• **Exercise 6:** Consider a function $f(a, b)$ of two strictly positive integer variables that satisfies $f(1, 1) = 2$ and such that for every a, b the following holds:

$$f(a+1, b) = f(a, b) + 2(a+b) \qquad \text{and} \qquad f(a, b+1) = f(a, b) + 2(a+b-1)\,.$$

Prove that for every a, b we have $f(a, b) = (a+b)^2 - (a+b) - 2b + 2$.

Solution (with PMI Two-dimensional): For all $a, b \in \mathbb{N}$ with $a, b \geqslant 1$, let $P(a, b)$ be the requested equality for $f(a, b)$. The base case $P(1, 1)$ is easy to check because by definition we have $f(1, 1) = 2$. For the first induction step, fix $b = 1$ and suppose that $P(a, 1)$ is true for some $a \geqslant 1$. We now prove $P(a+1, 1)$:

$$\begin{aligned} f(a+1, 1) &= f(a, 1) + 2(a+1) \\ &= \left[(a+1)^2 - (a+1) - 2 + 2\right] + 2(a+1) \\ &= \left[(a+1)+1\right]^2 - \left[(a+1)+1\right]. \end{aligned}$$

For the second induction step, fix $a \geqslant 1$ and suppose that $P(a, b)$ is true for some $b \geqslant 1$. We now prove $P(a, b+1)$:

$$\begin{aligned} f(a, b+1) &= f(a, b) + 2(a+b-1) \\ &= \left[(a+b)^2 - (a+b) - 2b + 2\right] + 2(a+b-1) \\ &= (a+b+1)^2 - (a+b+1) - 2(b+1) + 2. \end{aligned}$$

• **Exercise 7:** Prove that for all natural numbers n, k such that $k \leqslant n$ the binomial coefficient $\binom{n}{k}$ is a natural number. You can make use of the known formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. (Hint: Apply PMI Complete.)

Solution (with PMI Sum of variables): The base case is clear because $\binom{0}{0} = 1$. Reasoning with PMI Complete, suppose that $\binom{n}{k} \in \mathbb{N}$ holds for all $n, k \in \mathbb{N}$ with $k \leqslant n$ and $0 \leqslant n + k \leqslant m$. We are going to prove that $\binom{n}{k} \in \mathbb{N}$ holds whenever $n, k \in \mathbb{N}$ with $k \leqslant n$ and $n + k = m + 1$. If $k = 0$, then we have $\binom{n}{k} = 1$. If k (and hence n) are at least 1, then we can apply the formula in the statement: since $(n-1) + (k-1) \leqslant m$ and $(n-1) + k \leqslant m$, the two binomial coefficients on the right-hand side are natural numbers and hence so is their sum.

• **Exercise 8:** For all finite subsets F of $\mathbb{N}$, prove that the number of subsets of F equals $2^{\#F}$.

Solution (with PMI Partition): Consider the partition given by the cardinality. With cardinality 0 we only have the empty set: this has $2^0 = 1$ subsets, namely only the empty set. Suppose that the assertion holds for all sets of cardinality n, and let us prove the assertion for any given set F of cardinality $n + 1$. Mark one element f of F, and call F' the complement of f. The subsets of F not containing f are exactly the subsets of F': since F' has cardinality $\#F - 1$, there are $2^{\#F-1}$ such subsets. A subset of F containing f corresponds to a subset of F' if we remove the element f: again, there are $2^{\#F-1}$ such subsets. In total, the set F has $2^{\#F-1} + 2^{\#F-1} = 2^{\#F}$ subsets.

• **Exercise 9:** Prove that for all $n \in \mathbb{N}$ we have

$$(-1)^n = \begin{cases} 1 & for\ n\ even\,; \\ -1 & for\ n\ odd\,. \end{cases}$$

Solution (with PMI Jumps): The two base cases $n = 0$ and $n = 1$ are clear because $(-1)^0 = 1$ and $(-1)^1 = -1$. We can also easily compute $(-1)^2 = 1$. Suppose that n is even and that the statement holds for n. Then we have $(-1)^{n+2} = (-1)^n \cdot (-1)^2 = 1 \cdot 1 = 1$ and hence the statement is proven for $n + 2$ (which is even). Similarly, suppose that n is odd and that the statement holds for n. Then we have $(-1)^{n+2} = (-1)^n \cdot (-1)^2 = (-1) \cdot 1 = -1$ and hence the statement is proven for $n+2$ (which is odd).

Solution (with a case distinction in the induction step of PMI Classic): The base case $n = 0$ is clear because $(-1)^0 = 1$. Now suppose that the statement holds for n. If n is even, then we have $(-1)^n = 1$ and hence $(-1)^{n+1} = (-1)^n \cdot (-1) = 1 \cdot (-1) = -1$, which proves the statement for the odd number $n + 1$. If n is odd, then we have $(-1)^n = -1$ and hence $(-1)^{n+1} = (-1)^n \cdot (-1) = (-1) \cdot (-1) = 1$, which proves the statement for the even number $n + 1$.