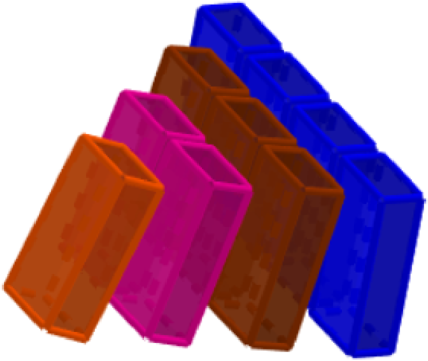
Visualizations for the Principle of Mathematical Induction



The Principle of Mathematical Induction (PMI)

PMI Classic

Consider statements $P(n)$ for $n \in \mathbb{N}$. Suppose that $P(0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ the statement $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

Exercise: Prove that for every $n \in \mathbb{N}$ the number $n^3 - n$ is a multiple of 3.

Visualization of PMI Classic

The PMI is usually illustrated by a row of falling dominoes:

- Consider infinitely (countably) many dominoes standing on end, arranged in a half-line extending infinitely to the right.
- The $(n+1)$th domino represents $P(n)$.
  Proving the truthfulness of $P(n)$ means that the corresponding domino falls to the right.

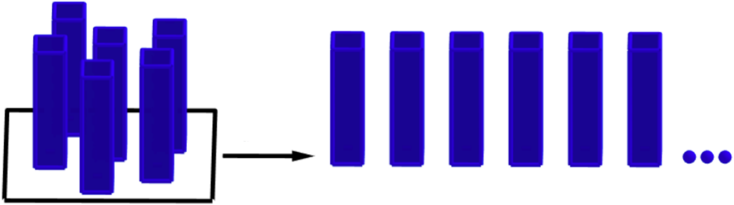
- **The base case:** Push the first domino as to make it fall. This starts the chain reaction. Without a push the dominoes keep standing.
- **The induction step:** If one domino falls, then its right-hand neighbor falls as well. This guarantees that the chain reaction includes all dominoes in the row (eventually each domino will fall). A missing induction step can be visualised by a row of dominoes that at some point is not tight (there is too much space between a domino and the next one, and hence the chain reaction does not propagate).

The PMI on a countable set

PMI Countably Infinite

If the set of statements is countably infinite, then it suffices to label its elements with the natural numbers to reduce to the situation of PMI Classic.

- We are arranging the dominoes in a row.



Examples for PMI Countably Infinite:

If we consider the set of even non-negative integers, then typically we choose 0 as first element, 2 as second, 4 as third, and so on (for the odd non-negative integers we would choose 1 as first element, 3 as second, 5 as third, and so on).

If we have the set of integers smaller than or equal to -5, then it is natural to take -5 as as first element, -6 as second, -7 as third, and so on.

If we have the set of all integers, then we can order these as follows:

$$0, 1, -1, 2, -2, 3, -3 \ldots$$

PMI Different Start

A special case of PMI Countably Infinite:

PMI Different Start

Let $n_0 \in \mathbb{N}$, and consider statements $P(n)$ for $n \in \mathbb{N}$ with $n \geqslant n_0$. Suppose that $P(n_0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ with $n \geqslant n_0$ the statement $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$ with $n \geqslant n_0$.

Exercise: Prove that for all natural numbers $n \geqslant 4$ we have $n \cdot (n-1) \cdots 2 \cdot 1 > 2^n$.

Visualization of PMI Different Start

- Think of a row of dominoes indexed by $\mathbb{N}$ (by considering some additional statements), and push the domino corresponding to n_0: the first dominoes stay untouched, the others will fall.
- Ignore the first dominoes: these could either fall if pushed (true statements) or they are fixed (false statements).

PMI Finite

PMI Finite

Let S be a non-empty finite set, and consider statements $P(s)$ for $s \in S$. We label the elements of S with the natural numbers from 0 to $c - 1$, where c is the cardinality of S. Suppose that $P(0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ with $0 \leqslant n < c - 1$ the statement $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(s)$ holds true for all $s \in S$.

Exercise: Prove that for all integers n in the range from 20 to 50 the binomial coefficient $\binom{30}{n-20}$ is strictly positive.

- Visually, the row of dominoes is finite: after finitely many steps all dominoes have fallen and the chain reaction stops.

Further variants of the PMI can be combined with PMI Countably Infinite or PMI Finite.

The complete mathematical induction

PMI Complete

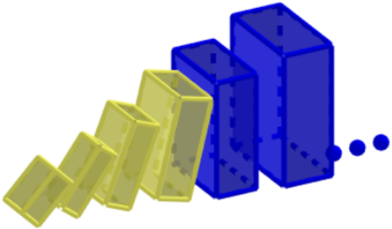
Consider statements $P(n)$ for $n \in \mathbb{N}$. Suppose that $P(0)$ is true (this is the base case). Suppose that for every $n \in \mathbb{N}$ the collection of statements $P(0)$ to $P(n)$ implies $P(n+1)$ (this is the induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

The induction step is easier to prove because we can make use of any statement from $P(0)$ to $P(n)$. Often we only need a fixed amount of previous statements, for example $P(n)$ and $P(n-1)$.

Further variants of the PMI can be combined with PMI Complete.

Visualization of PMI Complete

- Consider dominoes of growing size. The induction step means that the first dominoes together have enough elain to make the next domino fall.



Exercise: Prove that the n-th Fibonacci number equals

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

The backwards mathematical induction

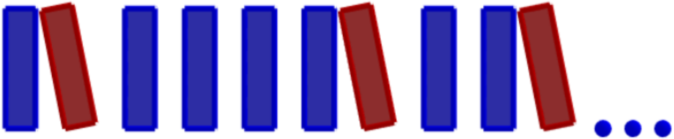
PMI Backwards

Consider statements $P(n)$ for $n \in \mathbb{N}$. Suppose that $P(n)$ is true for all $n \in S$, where S is an infinite subset of $\mathbb{N}$ (this is an infinite set of base cases). Suppose that for every $n \in \mathbb{N}$ with $n > 0$ the statement $P(n)$ implies $P(n-1)$ (this is the backward induction step). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

We are doing infinitely many applications of PMI Finite (each statement is proven multiple times).

Visualization of PMI Backwards

- Push to the left all dominoes corresponding to the elements of S: the chain reaction propagates to the left.



Exercise: Let $n \in \mathbb{N}$ with $n \geqslant 1$. Prove the inequality between arithmetic and geometric mean of n strictly positive real numbers:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \ldots a_n}\,.$$

(Hint: Prove the inequality by induction for all n that are powers of 2.)

Two-dimensional induction

PMI Two-dimensional

Consider statements $P(a, b)$ for $a, b \in \mathbb{N}$. Suppose that $P(0, 0)$ is true (this is the base case). Suppose that, if $P(a, 0)$ is true for some $a \in \mathbb{N}$, then $P(a+1, 0)$ is also true (this is the first induction step). Suppose that, if $P(a, b)$ is true for some $a, b \in \mathbb{N}$, then $P(a, b+1)$ is also true (this is the second induction step). Then $P(a, b)$ holds true for every $a, b \in \mathbb{N}$.

With the base case and the first induction step one proves $P(a, 0)$ for all $a \in \mathbb{N}$ (PMI Classic). The second induction step then allows to prove $P(a, b)$ for any fixed a and for any $b \in \mathbb{N}$ (infinitely many PMI Classic).

PMI Two-dimensional generalizes to finitely many variables.

Exercise: Consider a function $f(a, b)$ of two strictly positive integer variables that satisfies $f(1, 1) = 2$ and such that for every a, b the following holds:

$$f(a+1, b) = f(a, b) + 2(a+b)$$

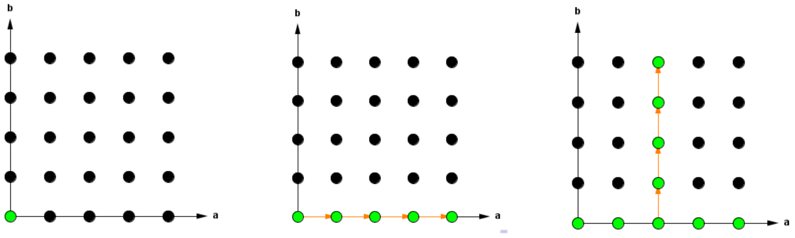
$$f(a, b+1) = f(a, b) + 2(a+b-1)\,.$$

Prove that for every a, b we have

$$f(a, b) = (a+b)^2 - (a+b) - 2b + 2\,.$$

Visualizing PMI Two-dimensional

- Mark the point (a, b) in the plane as soon as $P(a, b)$ is proven.
- Mark $(0, 0)$ because of the base case, and then the by first induction step all points on the a-axis.
- By the second induction step the marking propagates upwards from $(a, 0)$. It propagates to all points (a, b).



PMI Sum of variables

PMI Sum of variables

Consider statements $P(a, b)$ for $a, b \in \mathbb{N}$. Suppose that $P(0, 0)$ is true (this is the base case). Suppose that, if for some $n \in \mathbb{N}$ the statement $P(a, b)$ is true whenever $a+b = n$, then the statement $P(a, b)$ is true whenever $a+b = n+1$ (this is the induction step). Then $P(a, b)$ holds true for every $a, b \in \mathbb{N}$.

Consider statements $Q(n)$ consisting of all $P(a, b)$ with $a + b = n$, and apply PMI Classic.

Exercise: Prove that for all natural numbers n, k such that $k \leqslant n$ the binomial coefficient $\binom{n}{k}$ is a natural number. You can make use of the known formula

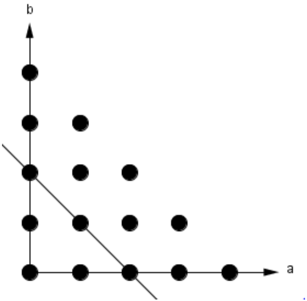
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

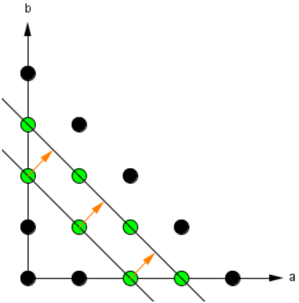
(Hints: While doing the induction in one variable, apply PMI Complete. The fact that the set of cases is a subset rather than the whole of $\mathbb{N}^2$ will not matter in the proof.)

PMI Sum of variables generalizes to finitely many variables.

Visualizing PMI Sum of variables

- The statement $Q(n)$ corresponds to the $(n+1)$th finite diagonal of the first quadrant.
- The chain reaction propagates from one diagonal to the next.





The general framework for grouping statements

PMI Partition

Let S be a set, and consider statements $P(s)$ for $s \in S$. Partition S into countably many subsets T_n with $n \in \mathbb{N}$. Suppose that $P(s)$ is true for all $s \in T_0$ (this is the base case). For all $n \in \mathbb{N}$ suppose that, if $P(s)$ is true whenever $s \in T_n$, then $P(s)$ is true whenever $s \in T_{n+1}$ (this is the induction step). Then $P(s)$ holds true for every $s \in S$.

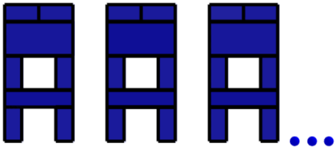
Apply PMI Classic to n.

One could consider a finite partition, and apply PMI Finite.

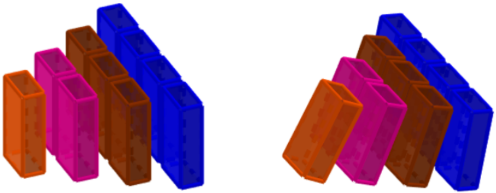
Exercise: For all finite subsets F of $\mathbb{N}$, prove that the number of subsets of F equals $2^{\#F}$.

Visualization of PMI Partition

- Consider domino towers (of growing size for PMI Complete). The towers completely fall apart in the process, i.e. all their dominoes fall.



- More generally, consider arrangements of dominoes: if all dominoes in an arrangement fall, then all dominoes in the next one fall.



Larger induction step

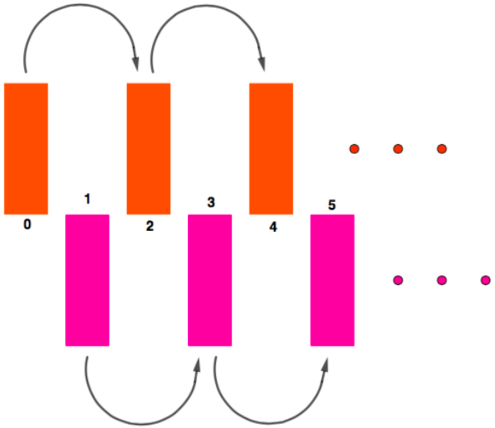
PMI Jumps

Consider statements $P(n)$ for $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ with $k \geqslant 1$. Suppose that the statements $P(0)$ up to $P(k-1)$ are true (we have k base cases). Suppose that for every $n \in \mathbb{N}$ the statement $P(n)$ implies $P(n+k)$ (in the induction step we jump k steps ahead). Then $P(n)$ holds true for all $n \in \mathbb{N}$.

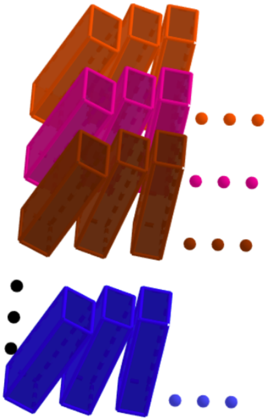
The set $\mathbb{N}$ is partitioned into k subsets, according to the remainder after division by k. Apply k times PMI Countably infinite.

Visualization of PMI Jumps

- PMI Jumps with $k = 2$ has two inductions in its structure, one for the even numbers and one for the odd numbers.



- Visualize PMI Jumps with k rows of falling dominoes: hitting the next domino in the row means jumping k steps ahead in the usual arrangement.



Case distinction on the proof of the induction step

An alternative to PMI Jumps is doing a **case distinction in the proof of the induction step** of PMI Classic.
For PMI Jumps with $k = 2$ one gets the two cases "from even to odd" and "from odd to even".

Exercise: Prove that for all $n \in \mathbb{N}$ we have

$$(-1)^n = \begin{cases} 1 & \textit{for } n \textit{ even}\,; \\ -1 & \textit{for } n \textit{ odd}\,. \end{cases}$$

Exercise: Prove (PMI Jumps with $k = 4$, or four cases) the formula for the higher derivatives of the sinus function.

Visualization of the case distinction

- Not all dominoes fall down in the same way, the dominoes are not aligned (here the dominoes are seen from above):

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