Staircase Numbers

We call a positive integer a *staircase number* (or *polite number* [2]) if it can be decomposed as the sum of at least two consecutive positive integers. For example, we have

$$12 = 3 + 4 + 5 \qquad 13 = 6 + 7 \qquad 15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5$$

while 16 is not a staircase number. Notice that we can visualize each decomposition in the form of a staircase:

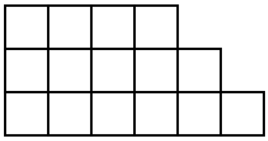

Figure 1: The decomposition $15 = 4 + 5 + 6$.

Which numbers are staircase numbers, and how many decompositions do we have?

As a warmup, notice that any odd number n is a staircase number because we can obtain it by summing the two integers closest to its half. For example, we have $21 = 10 + 11$. Also notice that if some integer $n > 3$ is divisible by 3, then it is a staircase number because we can write

$$n = \frac{n}{3} + \frac{n}{3} + \frac{n}{3} = \left(\frac{n}{3} - 1\right) + \frac{n}{3} + \left(\frac{n}{3} + 1\right)$$

where all summands are positive integers. For example, we have $21 = 6 + 7 + 8$.

In general, let n be a positive integer, and suppose that $d > 1$ is an odd divisor of n. Then we can write n as the sum of d copies of the integer $\frac{n}{d}$, and then we can add and subtract integers from these summands and get a sequence of consecutive integers:

$$n = \underbrace{\frac{n}{d} + \cdots + \frac{n}{d} + \frac{n}{d} + \frac{n}{d} + \cdots + \frac{n}{d}}_{d \text{ summands}} =$$

$$= \underbrace{\left(\frac{n}{d} - \frac{d-1}{2}\right) + \cdots + \left(\frac{n}{d} - 1\right) + \frac{n}{d} + \left(\frac{n}{d} + 1\right) \cdots + \left(\frac{n}{d} + \frac{d-1}{2}\right)}_{d \text{ summands}} .$$

If the smallest integer in the above decomposition, namely $\frac{n}{d} - \frac{d-1}{2}$, is at least 1, then we have obtained a valid decomposition of n as a staircase number with an odd number of terms. For example, considering $n = 15$ and $d = 3$, we obtain $15 = 4 + 5 + 6$.

If $\frac{n}{d} - \frac{d-1}{2} = 0$, then we can remove this summand and get a valid decomposition of n as a staircase number with an even number of terms. For example, $n = 10$ and $d = 5$ gives

$$10 = 2 + 2 + 2 + 2 + 2 = 0 + 1 + 2 + 3 + 4 = 1 + 2 + 3 + 4 .$$

If $\frac{n}{d} - \frac{d-1}{2} = -s$ for some positive integer s, then we can remove from the decomposition all integers from $-s$ to s included (this does not alter the total sum) and then we obtain a valid decomposition of n as a staircase number with an even number of terms. For example, considering $n = 14$ and $d = 7$, we obtain

$$14 = 2 + 2 + 2 + 2 + 2 + 2 + 2 = (-1) + 0 + 1 + 2 + 3 + 4 + 5 = 2 + 3 + 4 + 5 .$$

So if $d > 1$ is an odd divisor of n, then we have shown that n is a staircase number and we have explicitly constructed a decomposition.

It is not difficult to show that if we start with a different odd divisor of n greater than 1, then we obtain a different decomposition by applying the above construction. We deduce that we have at least as many decompositions as the number of odd divisors which are greater than 1. These are all the possible decompositions, as we can see by reversing the above procedure. Indeed, if we have a valid decomposition with an odd number $d > 1$ of consecutive integers, then this is exactly the decomposition which we obtain from the above construction starting from d. On the other hand, if we have a valid decomposition with an even number of terms, and whose smallest summand is s, then we can insert summands as to have a sequence of an odd number $d > 1$ of consecutive integers starting from $-(s-1) \leq 0$. Then this and also the initial decomposition that we had is what we obtain from the above construction starting from d. In summary, we have proven Sylvester's Theorem:

> *The only positive integers which are not staircase numbers are the powers of* 2*. For a staircase number there are as many decompositions (a decomposition being the sum of at least two consecutive positive integers) as the number of odd divisors greater than* 1.

Notice that triangular numbers[1] are in particular staircase numbers. As further reading we recommend [1], where the authors determine those triangular numbers having

[1] A triangular number is a positive integer which is the sum of all integers from 1 to n for some positive integer n. For example $3 = 1 + 2$ and $6 = 1 + 2 + 3$ are triangular numbers.

exactly one decomposition as staircase numbers. Apart from these very special staircase numbers, the other ones are called *trapezoidal numbers* because they are the difference of two triangular numbers and hence they form visually an isosceles trapezoid, for example $12 = (1 + 2 + 3 + 4 + 5) - (1 + 2) = 15 - 3$.

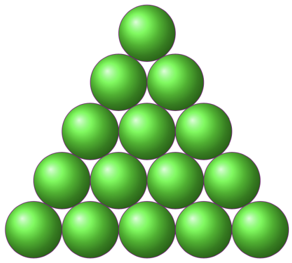

Figure 2: The triangular number 15, with on top the triangular number 3.

Exercises for the reader

1. Prove that distinct odd divisors $d > 1$ of some positive integer n give rise (with the construction that we have outlined) to different decompositions of n as a sum of positive consecutive integers.

2. Consider the decomposition of a positive integer $n > 1$ as the product of powers of distinct primes (this is the prime factorization where we have multiplied together equal prime numbers). Prove a formula for the number of odd divisors of n greater than 1 which involves the exponents of the prime powers in the factorization.

References

[1] Chris Jones and Nick Lord, *Characterising Non-Trapezoidal Numbers*, The Mathematical Gazette Vol. 83, No. 497 (Jul., 1999), pp. 262-263.

[2] Wikipedia contributors. *Polite number*. Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Polite_number`, retrieved November 27, 2020.

Solutions to the exercises for the reader

1. By reversing the construction as we have done in the article we see that the integer d is determined by the decomposition (consider separately the case of a decomposition with an even or an odd number of summands).

2. Let $p_1^{e_1}, \ldots, p_r^{e_r}$ be the odd prime powers appearing in the prime factorization. Then the positive odd divisors d of n are exactly those numbers of the form $d = p_1^{d_1} \cdot \cdots \cdot p_r^{d_r}$ where $0 \leq d_i \leq e_i$ for all i from 1 to r. This gives

$$(d_1 + 1) \cdot (d_2 + 1) \cdot \cdots \cdot (d_r + 1)$$

   odd positive divisors, to which we have to remove 1 in order to exclude the divisor $d = 1$.