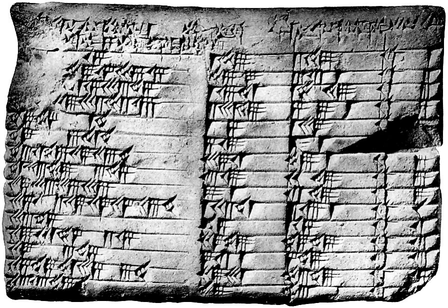
The decimal analogue of Plimpton 322



Plimpton 322 (P322)

- The famous Babylonian clay tablet Plimpton 322 dates back almost 4000 years ago.
- The tablet contains a table of numbers (written in the typical cuneiform script and expressed with digits in base 60).
- P322 can be seen as a list of Pythagorean triples, and it has probably been used for computations as a trigonometric table.

The purpose of P322

- The displayed numbers describe 15 right triangles with rational side lengths. Thus (up to rescaling) P322 is **a list of primitive Pythagorean triples.**
- The smallest angle in these right triangles is distributed quite well between 30 and 45 degrees (the following values are rounded):

$$44.8;\ 44.3;\ 43.8;\ 43.3;\ 42.1;\ 41.5;\ 40.3;$$

$$39.8;\ 38.7\,;\ 37.4;\ 36.9;\ 35.0;\ 33.9;\ 33.3;\ 31.9\,.$$

- P322 then allows to approximate non-skinny right triangles with one of the list for computational purposes, and hence **it can be used as a trigonometric table.** It is an exact trigonometric table (the numbers are not rounded).

Base 60 versus base 10

The mathematics beyond P322 is related to the base 60: for didactic purposes we construct its analogue in base 10.
This is not just writing the given numbers in base 10!

We need rational numbers with finitely many digits: in base 10 (respectively, 60), this means that the minimal denumerator divides a power of 10 (respectively, 60).

The decimal analogue of P322

L	H	L^2	n_L	n_H	row #
$\frac{3}{4}$	$\frac{5}{4}$	$\frac{9}{16}$	3	5	1
$\frac{39}{80}$	$\frac{89}{80}$	$\frac{1521}{6400}$	39	89	2
$\frac{369}{800}$	$\frac{881}{800}$	$\frac{136161}{640000}$	369	881	3
$\frac{399}{1600}$	$\frac{1649}{1600}$	$\frac{159201}{2560000}$	399	1649	4
$\frac{9}{40}$	$\frac{41}{40}$	$\frac{81}{1600}$	9	41	5

Comparison to the original P322

- The displayed numbers are not those of P322 (with the exception of the first row) because they are constructed by working in base 10 rather than in base 60.
- We write numbers in the decimal system (rather than in base 60 and in cuneiform script).
- Further changes w.r.t. the original P332: We have added the first two columns and changed the headings. We have chosen to write rational numbers as reduced fractions rather than using their digital expression.

Description of (the decimal) P322

- We have a table consisting of **rational numbers**. These have **finitely many digits**, i.e. their minimal denumerator divides a power of 10 (of 60 in the original P322).
- **Column description:** We have the very important rational numbers L and H. Then we have the square of L, the minimal numerator of L, the minimal numerator of H, and the row number.

The numbers L and H

- They are strictly positive rational numbers.
- They are the short leg and the hypotenuse of a right triangle in which the long leg equals 1, i.e. they satisfy $L < 1$ and the identity

$$L^2 + 1^2 = H^2 .$$

- They have the same minimal denominator (this follows from the above identity).
- They have finitely many digits, i.e. their minimal denominator divides a power of 10 (of 60 in the original P322).

The right triangle corresponding to L and H

Consider the right triangle with rational side lengths

$$(L, 1, H)\,.$$

If we rescale it so that the side lengths are coprime integers, then we get the primitive Pythagorean triple

$$(n_L, d, n_H)\,,$$

where d is the (common) minimal denominator of L and H. The middle number of the triple divides a power of 10 (of 60 in the original P322).

Example:

$$\left(\frac{3}{4}, 1, \frac{5}{4}\right) \qquad \rightsquigarrow \qquad (3, 4, 5)$$

Reconstructing P322

- Babylonians were looking for right triangles whose side lengths are rational numbers with finitely many digits (in base 60), and such that the long leg equals 1. In other words, they were looking for primitive Pythagorean triples such that the middle number divides a power of 60.
- To produce the decimal analogue of P322, we rely on the algorithm which is presented in [MW] for the construction of an extended version of Plimpton 322. Conjecturally, Babylonians used this or a similar algorithm to produce the numbers displayed in the tablet.

The algorithm: Part I

- List all regular integers (for base 10) from 1 to 60, i.e. all integers s in this range such that $s = 2^a \cdot 5^b$, where a, b are non-negative integers:

$$1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50.$$

- For every number s in the above list, find all regular integers r (for base 10) such that

$$s < r < (1 + \sqrt{2})s\,.$$

The choice of this parameter interval will be clear later.

The algorithm: Part II

- We are only interested in the rational number $\frac{r}{s}$, and removing all duplications we find:

s	r	r/s
1	2	2
4	5	5/4
5	8	8/5
16	25	25/16
25	32	32/25

- For all above values for $\frac{r}{s}$, compute the two strictly positive rational numbers

$$L := \frac{1}{2}\left(\frac{r}{s} - \frac{s}{r}\right) \qquad \text{and} \qquad H := \frac{1}{2}\left(\frac{r}{s} + \frac{s}{r}\right) .$$

The inequality $s < r$ ensures that $L > 0$.
The numbers $(L, 1, H)$ are side lengths of a right triangle because we have $L^2 + 1 = H^2$.
The inequality $r < (1 + \sqrt{2})s$ ensures that L is the short leg i.e. that $L < 1$.

The algorithm: Part III

- Compute L^2 and the minimal numerators n_L and n_H of L and H respectively.
- List the tuples (L, H, L^2, n_L, n_H) so that the first entry L (equivalently, H or L^2) is in decreasing order. Finally, add a row number to identify the tuples.

If we want to produce more right triangles, then we have to increase the upper bound for the parameter s: In base 60, we have more regular integers and taking $s < 60$ already gives 38 triangles.

Remark: In the original Plimpton 322 there are only right triangles such that the smallest angle is larger than 30 degrees. To obtain these, we can apply the algorithm and select those triangles such that $2L > H$.

Exercises

1. Show that if two rational numbers L and H satisfy $L^2 + 1 = H^2$, then they have the same minimal denominator.
2. Show that two strictly positive rational numbers r and s satisfy $r < (1 + \sqrt{2})s$ if and only if the number $L := \frac{1}{2}\left(\frac{r}{s} - \frac{s}{r}\right)$ is smaller than 1.
3. Use the appropriate row of the decimal analogue of Plimpton 322 to approximately compute the hypotenuse of the right triangle with leg lengths 3 and 13.

References

[K] Knuth, D., 1972. *Ancient Babylonian algorithms*. Commun. ACM15 (7), 671–677.

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