The Art Gallery Problem

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In an art gallery there are priceless paintings, and therefore the whole museum must constantly be under surveillance: How many cameras are needed? Such a question is called the Art Gallery Problem (which has been formulated as a mathematical problem in 1973).

All surveillance cameras are fixed, and they can see in any direction. We suppose that the cameras are as small as a point, and they can be installed for example in the very corner of a room, or at the edge between the ceiling and a wall. We also assume that the paintings hang on the walls without thickness, and that there are no further objects. Finally, we will suppose that the height of the ceiling is the same in the whole museum. Thus we have a two-dimensional problem, and to solve it we only have to consider the museum map and study its geometry (this could be a non-trivial task, see Figure 1).

Figure 1: Weisman Art Museum in Minneapolis

If the museum consists in one circular room, then clearly one camera suffices, and this camera can be installed anywhere. If there is only one room in the shape of a star, then again one camera suffices, but this time the camera must be installed around the center of the star and not, say, in one of the star's spikes. On the other hand, if the museum would consist in one room in the shape of the letter C, then one camera is not sufficient. The mathematics beyond the above examples is as follows: If the museum map is *convex*, then any point sees all the other points, and we can install just one surveillance camera anywhere we want (examples of convex figures are triangles, rectangles, and ellipses; not convex is a figure with some kind of hollow or hole, for example a dart or an annulus). If the museum map is *star-shaped*, then this means that there is one figure point that sees all the other points, so again one surveillance camera suffices, but we may have to choose carefully where to install it. Any convex object is in particular star-shaped, and stars are of course star-shaped: one further example is an L-shaped object. However, there are polygons which are not star-shaped (see Figure 2).

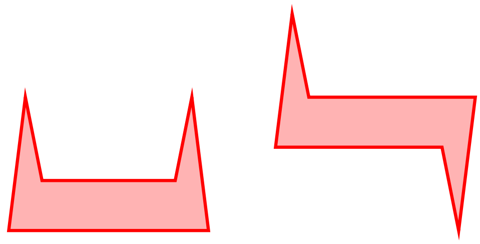


Figure 2: Examples of polygons which are not star-shaped

One general result is the following: Consider a museum map that is a polygon like the ones we are most used to (i.e. planar, closed, and not self-intersecting). A sufficient and sometimes necessary amount of cameras is *the quotient after division by 3 of the number of polygon edges.* For example, a polygon with six edges may be convex (one camera suffices) or it may have spikes (two cameras are then necessary, see Figure 2). This result is from 1975 and is due to Václav Chvátal, however a more visual proof has been given by Steve Fisk in 1978 [1]. We leave it to the reader to argument that the given amount of cameras is sometimes necessary (see Exercise 3). To show that the cameras are sufficient, Fisk's idea is triangulating the museum map, i.e. dividing it into triangles, and then installing surveillance cameras into some of the triangles' vertices. More precisely:

- Our polygon can be triangulated without adding extra vertices: This means that we can draw some diagonals so that the polygon becomes the union of triangles. Note, the diagonals that we choose should not intersect.
- We are going to put the cameras in the triangles' vertices, and remark that one vertex for each triangle suffices to ensure the surveillance. We may color all vertices with three colors in such a way that every triangle has vertices of different colors.
- We choose a color which occurs the least amount of times and place the cameras in all vertices with that color. Notice that the total number of vertices is the same as the number of polygon edges. Thus there is always one color that occurs at most as many times as the quotient after division by 3 of the number of polygon edges.

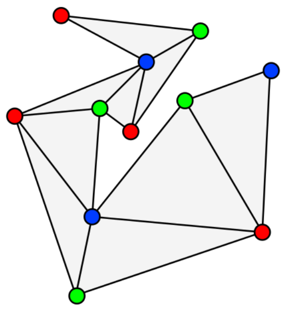


Figure 3: A triangulated polygon with 11 vertices.

In practice, one could make use of a computer to take care of all possible triangulations of the museum map, and more generally for solving the Art Gallery Problem in an optimal way for a specific example. For some variants of the Art Gallery Problem we refer to [2].

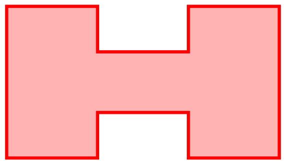
References

[1] Martin Aigner and Günter Ziegler (2009). *Proofs from THE BOOK* (4th ed.). Berlin, New York: Springer-Verlag. ISBN 978-3-642-00855-9.

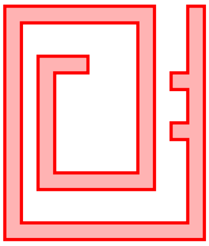
[2] Wikipedia contributors. *Art gallery problem.* Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Art_gallery_problem`, retrieved November 12, 2018.

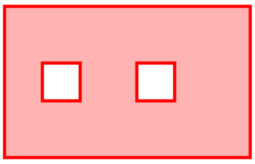
Exercises

1. Consider the Art Gallery Problem for a museum in shape of an H (there are, say, two rooms and a corridor). How many surveillance cameras are needed? You should be able to argument that the amount of cameras is necessary and sufficient.



2. Consider the Art Gallery Problem for the two museums depicted here. How many surveillance cameras would you install, and where would you place them?





3. Consider the Art Gallery Problem for a simple polygon (planar, closed and not self-intersecting) with n edges. Find examples for which one needs as many surveillance cameras as the quotient of n after division by 3.

Solutions to the exercises

1. Two cameras are sufficient: we can place one camera in the left room, and one camera in the right room at the level of the corridor. The first camera takes care of the left room, while the second camera takes care both of the right room and the corridor. One camera does not suffice: this is because some of the rooms' corners are only seen from inside the room itself.

2. *First museum:* Suppose that the museum entrance is on the upper right. One needs at least two cameras in the first corridor, in front of the two recesses. We also need one camera in the second corridor: we may place it at the end, so that it will also cover the third corridor. Placing a camera at the corner of the fourth and fifth corridor will cover both of them, and similarly for the sixth and the seventh corridor. Finally, we need a camera for the eight corridor. We thus have a total of six cameras. *Second museum:* Three cameras suffice: Place one in the upper left corner, one in the bottom right corner, and one in the middle.

3. For $n \leqslant 5$ the assertion is clear because at least one camera is needed. For $n = 6$ it suffices to consider museum maps with two spikes (as in Figure 2). More generally, if $n = 3t$ for some $t \geqslant 2$, then we can draw a museum map with t spikes, and such that one camera is needed for each spike (see the figure below). If $n = 3t + 1$ or $n = 3t + 2$, then we can simply modify the museum map for $n = 3t$ by inserting one or two edges (for example by cutting off the top of one or two spikes).