The ABCD of cyclic quadrilaterals

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For any triangle the following holds: all the three vertices lie on one same circle (called *circumcircle*). Let's see what happens for quadrilaterals. Is there a circumcircle or, in other words, do all four vertices lie on one same circle? If this is the case, the quadrilateral is called *cyclic*.

If we take three of the four vertices, they lie on one same circle (which is uniquely determined) because the three points are not aligned. In order for the quadrilateral to be cyclic, the fourth point must lie on that circle as well. This is the case for example for rectangles and isosceles trapezoids, as it can easily be seen by exploiting the symmetry of these figures. An example of quadrilateral without a circumcircle is a rhombus which is not a square (because the diagonals would be two chords which cross perpendicularly and bisect one another, so they would both be diameters, which is impossible because they have distinct lengths).

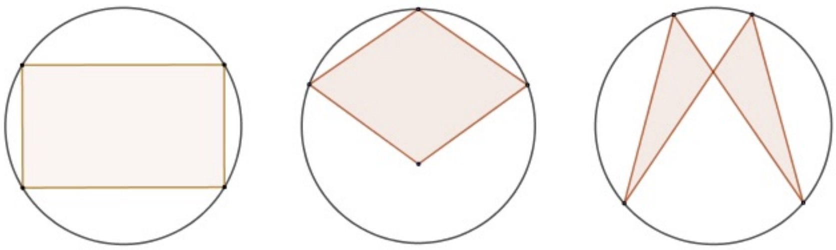

Figure 1: A rectangle, a rhombus and an antiparallelogram

The non-convex quadrilaterals that admit a circumcircle must be crossed (we leave it up to you to prove this fact). For example an antiparallelogram (namely a crossed quadrilateral such that the two opposite sides and the two crossing sides have equal length) has the same vertices of an isosceles trapezoid, so it is cyclic.

There are various properties for a convex quadrilateral which are equivalent to being cyclic, we list below the basic ones.

Notation: We will denote by $ABCD$ the four vertices of the quadrilateral (in cyclic order), and by $\alpha, \beta, \gamma, \delta$ the interior angles.

1. *A convex quadrilateral is cyclic if and only if the sum of a pair of opposite angles is* $180°$.
   For the direct implication, consider the circumcircle of $ABCD$, and the angles β and δ. The angles at the center corresponding to β and δ are 2β and 2δ, and they add up to $360°$. We deduce that $\beta + \delta = 180°$.

   For the converse implication, consider the circle passing through A, B, C and the two halfplanes H_B and H_D determined by the line AC and containing the point B and D respectively. The arc AC in H_B consists of all points P in H_B such that $\widehat{APC} = \beta$ hence the arc AC in H_D consists of all points P in H_D such that $\widehat{APC} = 180° - \beta = \delta$: this arc must contain D, so all points A, B, C, D lie on one same circle.

2. *A convex quadrilateral is cyclic if and only if the angle between a side and a diagonal is equal to the angle between the opposite side and the other diagonal.*

   Consider the circle containing A, B, C. Since C, D lie in the same halfplane with respect to the line AB, the following holds: the angles $\widehat{ACB}$ and $\widehat{ADB}$ are the same if and only if the points C and D lie on one same circle having AB as chord. This proves the criterion because by renaming the vertices we may suppose that the angles in the property are $\widehat{ACB}$ and $\widehat{ADB}$.

3. *(Ptolemy's Theorem) A convex quadrilateral is cyclic if and only if the product of its diagonals equals the sum of the products of the pairs of opposite sides.* Set $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = p$, $BD = q$.

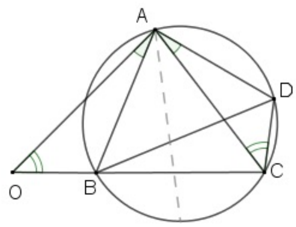

Figure 2: Illustration of the proof of Ptolemy's Theorem

The line which is symmetric to the line AD with respect to the bisector of $\widehat{BAC}$ contains exactly one point O such that $\widehat{BOA} = \widehat{DCA}$. We also have by symmetry $\widehat{OAB} = \widehat{CAD}$. So we get $\widehat{ABO} = \widehat{ADC}$: the first angle is $\widehat{CBO} - \beta$ and the second angle is δ. We deduce from the first characterization that the point O lies on the line BC if and only if $ABCD$ is cyclic.

In any case, the triangles $\triangle AOB$ and $\triangle ACD$ are similar, and hence also the triangles $\triangle OAC$ and $\triangle BAD$ are similar (because $\widehat{OAC} = \widehat{BAD}$ and $\frac{AO}{AC} = \frac{AB}{AD}$). We deduce that

$$OB = \frac{ac}{d} \quad \text{and} \quad OC = \frac{pq}{d}\,.$$

The formula in the statement is $pq = ac + bd$, which is equivalent to

$$\frac{pq}{d} = \frac{ac}{d} + b\,.$$

This can be restated as $OC = OB + BC$, so it is equivalent to the fact that the three points O, B, C are aligned. We have seen that this is equivalent to $ABCD$ being cyclic.

4. *A convex quadrilateral $ABCD$ is cyclic if and only if we have*

$$\overline{AE} \cdot \overline{EC} = \overline{BE} \cdot \overline{ED}\,,$$

where E is the point of intersection of the diagonals.

Since $\widehat{AEB} = \widehat{DEC}$, the equality holds if and only if the triangles $\triangle AEB$ and $\triangle DCE$ are similar and hence if and only if $\widehat{DBA} = \widehat{DCA}$. We conclude by the second characterization that this happens if and only if $ABCD$ is cyclic.

5. *A convex quadrilateral is cyclic if and only if the perpendicular bisectors of all four sides meet at one point.*

The perpendicular bisector of a chord of a circle passes through the center of the circle. Thus, the perpendicular bisectors of the sides of a cyclic quadrilateral meet at the center of the circumcircle.

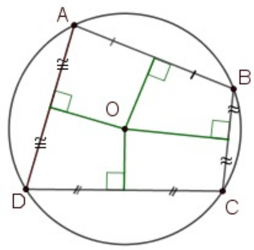

Figure 3: Illustration of the characterization with perpendicular bisectors

Now suppose that the perpendicular bisectors of all four sides meet at one point O. Then O is the center of the circle passing through A, B, C, and also the center of the circle passing through B, C, D. So the two circles coincide and contain all points A, B, C, D.

6. *A convex quadrilateral is cyclic if and only if it has maximal area among the quadrilaterals having its same side length.* Bretschneider's formula for the area of a general quadrilateral states that the area is

$$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cdot \cos^2\left(\frac{\alpha+\gamma}{2}\right)}$$

where s stands for the semi perimeter and α and γ are two opposite angles of the quadrilateral. The value for the area is maximal if and only if $\cos^2\left(\frac{\alpha+\gamma}{2}\right) = 0$, which means that $\alpha + \gamma = 180°$. We conclude by the first characterization that this happens if and only if $ABCD$ is cyclic.

The above list gives us six chances to prove that a convex quadrilateral is cyclic. Indeed, it suffices to prove the equivalent property. Moreover, we have also shown that convex cyclic quadrilaterals satisfy all above six properties. In fact, they have many more properties!

Questions for the reader

1. Show that a trapezoid is cyclic if and only if it is an isosceles trapezoid.

2. Prove that there are no non-convex and non-crossed cyclic quadrilaterals.

Solutions to the questions for the reader

1. Show that a trapezoid is cyclic if and only if it is an isosceles trapezoid.

   Solution. A trapezoid is isosceles if and only if a pair of opposite angles adds up to $180°$. By the first characterization this happens if and only if the trapezoid is cyclic. Another argument which uses symmetries: For an isosceles trapezoid, consider the circle containing A, B, C. The symmetry with respect to the perpendicular bisector of AB leaves the trapezoid and the circle invariant, and it maps C to D. We deduce that D is also on the circle and hence $ABCD$ is cyclic.

   □

2. Prove that there are no non-convex and non-crossed cyclic quadrilaterals.

   Solution. A non-convex and non-crossed quadrilateral cannot be cyclic because the vertex of the concave angle lies inside the triangle made by the other three vertices and the circumcircle of the triangle cannot contain such a point.

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