Four points, two distances

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If you take four distinct points on a plane they determine six edges, which in general may have different lengths. It is not possible that all edges have the same length (otherwise the points would be the vertices of a tetrahedron) but for some very special configurations the edges only have two distinct lengths. The "Four points, two distances" puzzle by the mathematician Peter Winkler is then the following:

> Find all configurations of four distinct points in the plane that determine exactly two distances.

You are encouraged to think of this problem before reading the solution below. There are six different configurations. . .

The proof is divided in several steps, and we recommend to follow it with pen and paper. We name the points A, B, C, D and we speak of the short/long distance between the points, and of the short/long edges.

Step 1

We prove that no three points are collinear. Suppose without loss of generality that A, B, C are collinear, with B between A and C. Then $\overline{AB} = \overline{BC}$ is the short distance while $\overline{AC}$ is the long distance. If $\overline{DA} = \overline{DC}$, then D lies on the perpendicular bisector of the segment AC. Since D is distinct from B, then $\overline{DA}$ must be the long distance. However, if that is the case, ACD is an equilateral triangle and $\overline{DB}$ is neither the short nor the long distance, which is impossible. If $\overline{DA} \neq \overline{DC}$, then suppose without loss of generality that $\overline{DA}$ is the short distance and $\overline{DC}$ is the long distance. Then D lies in the half-plane cut by the perpendicular bisector of AC and which contains A. In particular, $\overline{DB}$ is the short distance (being shorter than $\overline{DC}$), so we find that ABD is an equilateral triangle and that $\overline{DC}$ cannot be the long distance (the triangle ACD is not isosceles), contradiction.

Step 2

There are at least two short edges that share a vertex. Notice that there must be at least two short edges. Indeed, if there are five long edges then the configuration must be two equilateral triangles with a shared edge, and the remaining edge would have to be longer. If there are at least three short edges then some of them must share a vertex. To conclude we only have to exclude that there are exactly two short edges not sharing a vertex. Without loss of generality, let $\overline{AB}$ and $\overline{CD}$ be the two short edges. Then $\overline{AC} = \overline{BC}$ are long edges and hence C lies on the perpendicular bisector of AB. The same holds for D. Since, $\overline{AC} = \overline{AD}$ is the long edge (and $C \neq D$), the points C and D must lie symmetrically with respect to the line AB. But then $ABCD$ is a square where AB and CD are the diagonals, contradicting that $\overline{AC}$ is greater than $\overline{AB}$.

Step 3

There is a triangle with two short and one long edge. We proceed by the cases of how many long edges there are. Notice that by Step 2 there are at most 4 long edges. Moreover, we cannot have all short edges, otherwise the points would be the vertices of a tetrahedron and could not lie in the plane.
– If there is exactly one long edge, then either triangle it forms will suffice.
– If there are exactly two long edges, consider one of them. It is part of two triangles, at most one of which can contain the other long edge, so at least one of which has two short sides.
– If there are exactly three long edges then we have three short edges. Take two short edges that share a vertex (by Step 2), without loss of generality let them be $\overline{AB}$ and $\overline{BC}$. If the triangle ABC is equilateral, then all distances to the point D would have to be long, which is not possible. So the side $\overline{AC}$ is long and ABC is the triangle we are looking for.
– If there are exactly four long edges, then the two short edges share a vertex and the remaining edge of that implied triangle is long.

Step 4

Finding all solutions. Without loss of generality (by Step 3) suppose that ABC is a triangle such that $\overline{AB} = \overline{BC}$ are the short edges, and $\overline{AC}$ is the long edge. We may also suppose without loss of generality that D lies in

the half-plane cut by the perpendicular bisector of AC which contains C (possibly D lies on the perpendicular bisector). Then it is not possible that $\overline{AD}$ is short while $\overline{CD}$ is long. We are left with six cases:

Case	$\overline{AD}$	$\overline{BD}$	$\overline{CD}$
1	short	short	short
2	short	long	short
3	long	short	short
4	long	short	long
5	long	long	short
6	long	long	long

We now find the six possible configurations going through the six cases of the above table.

Case 1 (rhombus): The point D lies on the perpendicular bisector of AC and it has to be the reflection of the point B across the line AC. So we have a pair of equilateral triangles that share the edge BD.
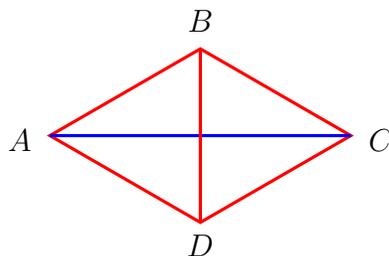
Case 2 (square): The point D lies on the perpendicular bisector of AC and it has to be the reflection of point B across the line AC. In this case we get a square with AC and BD as diagonals.
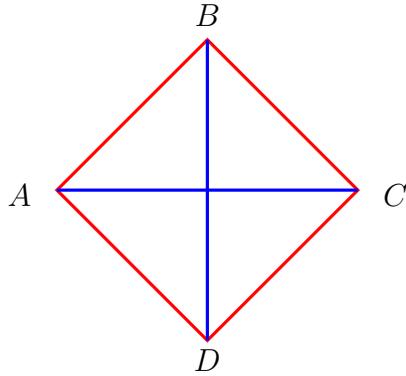
Case 3 (isosceles triangle): The point A lies on the perpendicular bisector of CD because the triangle ADC is isosceles. The triangle BCD is equilateral. The point B must then lie in the triangle ACD because $\overline{AB}$ is the short edge.
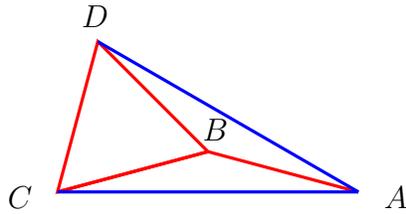
Case 4 (equilateral triangle): The triangle ACD is equilateral, and B is the middle point, having the same distances from A, C, and D.
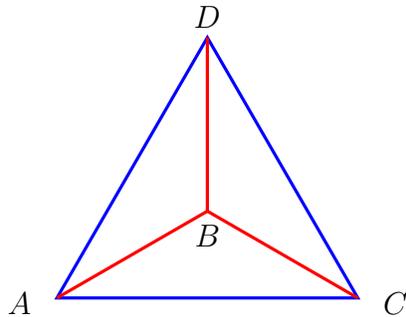
Case 5 (pentagon): The point D lies on the perpendicular bisector of AB, on the side of C (otherwise $\overline{DC}$ would be longer than the long edge $\overline{DB}$). The triangles ABD and ACD are isosceles and congruent, sharing a long edge. We deduce that the four points form an isosceles trapezoid with bases AD and BC. Since the trapezoid has three equal sides and the diagonal equals the long basis, we deduce that the four points are vertices of a regular pentagon.
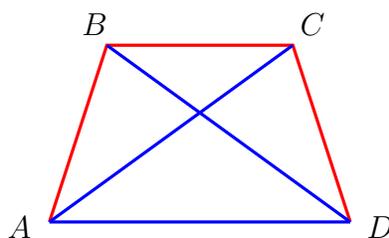
Case 6 (kite): Now ACD is an equilateral triangle. The points B and D lie on the perpendicular bisector of AC, and they lie on opposite sides with respect to AC (because $\overline{BD}$ is a long edge). The four points form a kite with diagonals AC and BD of equal length.
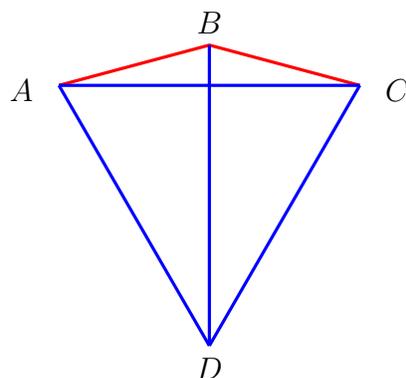
Acknowledgements

Questions for the reader

(1) Prove that the vertices of an isosceles trapezoid are vertices of a regular pentagon if the trapezoid has three equal sides and the diagonal equals the long basis.

Solution: Let AB be the long basis and DC be the short basis. Call α the angle at A. Since the long basis equals the diagonal we deduce that the angle $A\hat{D}B$ equals α. Call β the angle $B\hat{D}C$, so that the angle at D is $\alpha + \beta$. The sum of all angles in the trapezoid is then $2\pi = 4\alpha + 2\beta$. We also have the relation $\alpha + \beta = D\hat{C}B = \pi - 2\beta$ by considering the isosceles triangle DBC (where $B\hat{D}C = C\hat{B}D = \beta$). The above relations for α and β suffice to deduce that the angles in C and D (which is $\alpha + \beta$) equals $108°$. Thus the sides AC, DC, and CB are three consecutive sides of a regular pentagon.
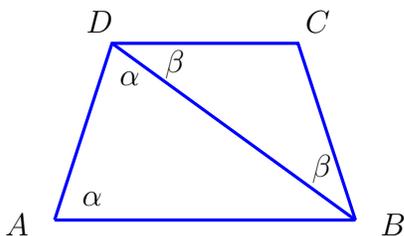
(2) For the six configurations determine the short length, supposing that the long length is 1.

Solution: It is straight-forward to compute the short length in the case "rhombus" ($\frac{\sqrt{3}}{3}$), "square" ($\frac{\sqrt{2}}{2}$), and "equilateral triangle" ($\frac{\sqrt{3}}{3}$). For "pentagon" we have the well-known ratio between diagonal and side, whose reciprocal is the requested length ($\frac{\sqrt{5}-1}{2}$). For "isosceles triangle" we can compute the height at C of the isosceles triangle ACD, and by Pythagora's theorem we find that the requested length is $\sqrt{2 - \sqrt{3}}$, which can be rewritten as $\frac{\sqrt{6}-\sqrt{2}}{2}$. For "kite" we may compute the diagonal BD minus the height of the equilateral triangle. Then by Pythagora's theorem we find that the requested length is $\sqrt{2 - \sqrt{3}}$, which can be rewritten as $\frac{\sqrt{6}-\sqrt{2}}{2}$.