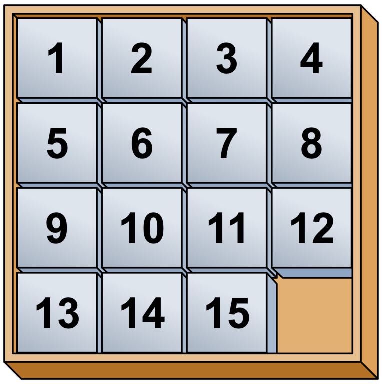
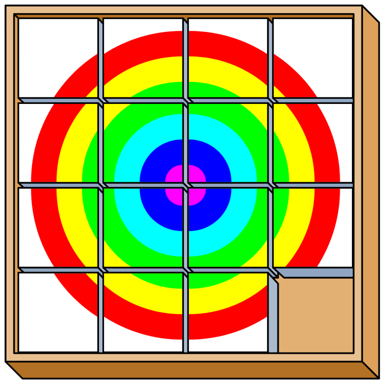
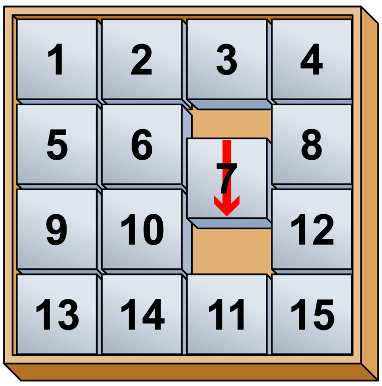
The 15 puzzle





An old classic among the mathematical puzzles is the 15-puzzle, which was invented at the end of the 19th century. On a 4×4 board 15 square tiles are arranged in a neat configuration. Usually the numbers from 1 to 15 are displayed, alternative versions show an image partitioned into 15 pieces. The tiles are then scrambled with the following rule: a tile adjacent to the empty slot can be moved either vertically or horizontally to the empty slot.



After the tiles have been scrambled, the game consists in recovering the initial configuration by moving the tiles with the same rule. It has been computed [2] that the smallest amount of moves needed to unscramble a configuration is at most 80, and that there are 17 configurations requiring 80 moves [1].

Solving the 15-puzzle

There are various strategies for solving the 15-puzzle. One can start by placing the tile 1 back to its position in the upper left corner, disregarding the position of the other tiles. Then the tile 1 will not be moved anymore through the rest of the game. Similarly one places 2 back to its position as second number in the first row, and this tile will not be moved anymore. To complete the first row we need a trick because tiles in a corner are hard to move.

To place 3 and 4 correctly, one places 4 in the third position of the first row, paying attention that by doing this 3 does not land in the upper right corner. Then one places

3 just below 4, and then the empty slot in the upper right corner. Finally, one lets 4 slide rightwards and 3 upwards to their correct positions.

We may similarly place the tiles 5,6,7,8 correctly in the second row. Hovever, for the last two rows we proceed differently because there is not enough space left. We place the remaining tiles correctly from left to right, so we first place 9 and 13 correctly. We use basically the same trick as above. We put 13 in the first place of the third row, paying attention that by doing this 9 does not land in the lower left corner. Then we place 9 right next to 13. Finally, one slides 13 downwards and 9 leftwards to their correct positions. We may similarly place 10 and 14.

We are left to place the numbers 11,12, and 15 around the lower right corner. If we make sure that 11 is in the correct position and the empty place is in the corner then the remaining tiles 12 and 15 must also be in their correct positions (this assertion can be deduced by studying the possible configurations) and hence the game is solved!

Notice that the strategy that we have outlined also holds for variants of the 15-puzzle made with square boards of greater size, or even with rectangular boards.

The possible configurations

We have plenty of ways to scramble the numbers from 1 to 15 on the 4×4 table with the given rule of the 15-puzzle. However, by this scrambling we do not obtain all the possible permutations of the numbers on the board. This can be seen as follows.

To every permutation we can associate a number, which we call its invariant. The invariant is either 1 or -1: half of the permutations have number 1 and half have number -1. Moreover, the invariant is 1 for the initial configuration and does not change with the admissible scrambling. We deduce that the admissible configurations must have number 1, and hence those with number -1 are impossible configurations. One can also show [3] that all permutations with invariant 1 are possible configurations for the 15-puzzle.

The number of possible configurations is then half of the number of permutations of 16 objects (we consider the empty square also as an object). So we may compute that there are more than ten thousand billions of possible configurations for the 15-puzzle (namely, the half of $16! = 16 \cdot 15 \cdot 14 \cdots 2 \cdot 1$).

Finally, we explain which permutations have number 1 and which have number -1. Each permutation of 16 objects can be obtained as a sequence of transpositions (a transposition is just the swap of two elements). We count the number of transpositions that we use for the given permutation. Moreover, we count the number of rows plus the number of columns of the empty square from the lower right corner. If the sum of those three numbers is even, the permutation has invariant 1, and if it is odd the invariant is -1. In particular, swapping two of the 15 numbers changes the invariant from 1 to -1 and conversely.

Question for the reader: At the end of our solution of the 15-puzzle we claimed that the tiles 12 and 15 must be in their correct positions. Can you prove this claim by considering the invariant?

References

[1] Korf, R. and Schultze, P. 2005. *Large-scale parallel breadth-first search.* In Proceedings of the 20th National Conference on Artificial Intelligence (AAAI-05), 1380–1385.

[2] Brungger, A., Marzetta,A., Fukuda,K. and Nievergelt,J. 1999. *The parallel search bench ZRAM and its applications.* Annals of Operations Research 90:45–63.

[3] Johnson, W. Woolsey and Story, W.E. 1879. *Notes on the "15" Puzzle*, American Journal of Mathematics, 2 (4): 397–404, doi:10.2307/2369492, ISSN 0002-9327, JSTOR 2369492.