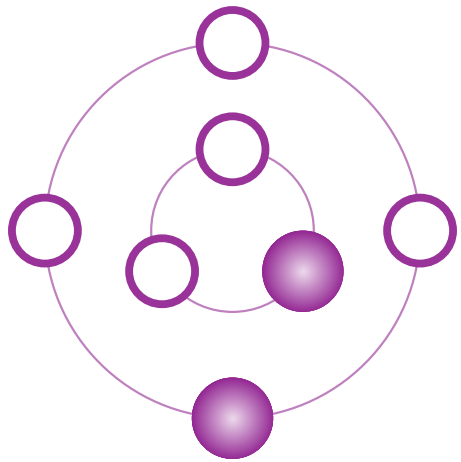
Chinese Remainder Theorem explained with rotations



Reference: Antonella Perucca, *The Chinese Remainder Clock*, The College Mathematics Journal, 2017, Vol. 48, No. 2, pp. 82-89.

One step-wise rotation

Consider a circle with one marked point on it.

- Fix some positive integer m. At every time unit, let the point move $\frac{1}{m}$-th of the circle clockwise.
- This phenomenon is periodic, the fundamental period is m.
- We identify the positions with the integers from 0 to $m-1$. For convenience, we set the initial position and 0 at the top.

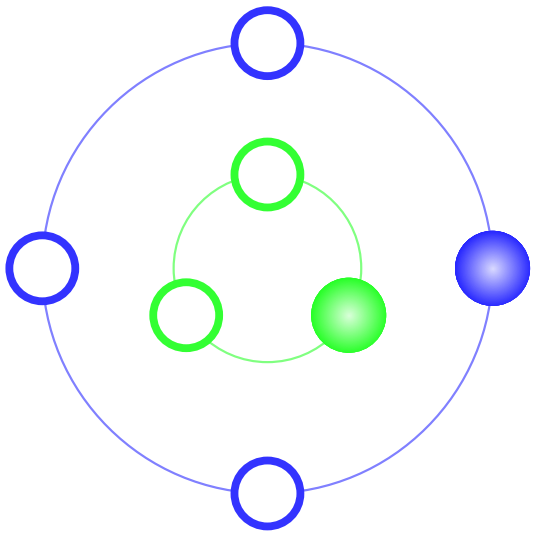
Example: $m = 3$, the position at time 0

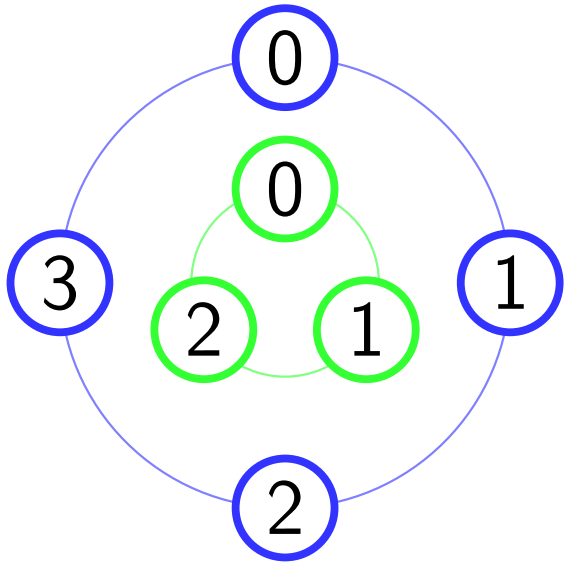
Two simultaneous rotations

Consider two simultaneous rotations with m_1 and m_2 steps.

- There are $m_1 \cdot m_2$ configurations for the pair of points.

Example: $m_1 = 3$ and $m_2 = 4$, the configuration at time 1



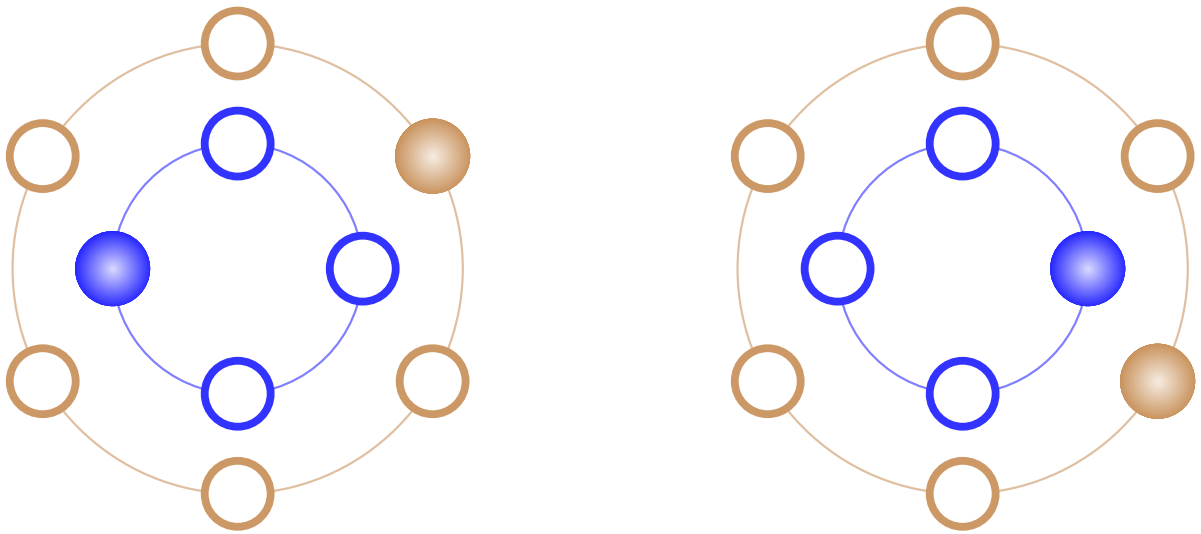


Impossible configurations

Consider two simultaneous rotations with m_1 and m_2 steps.

- In general, not all of configurations occur because the rotations are simultaneous (this is evident if $m_1 = m_2$).

Example: $m_1 = 4$ and $m_2 = 6$, the configuration at time 7 and an impossible configuration



Fundamental period for two rotations

- **The fundamental period is the least common multiple of the single periods.**
  Indeed, the fundamental period is the smallest positive time at which both points are back to the initial position, namely the least common multiple of m_1 and m_2.
- Each configuration occurs at most once in every fundamental period (because getting the same configuration implies that both points did full turns).

We deduce:

- **The fundamental period is the number of occurring configurations.**
- Special case m_1 and m_2 coprime:
  The fundamental period is $m_1 \cdot m_2$, every configuration occurs.

Generalisation to several rotations

Consider simultaneous rotations with $m_1, m_2 \ldots, m_n$ steps.

- **The fundamental period is the least common multiple of the single periods.**
  Namely, the least common multiple of $m_1, m_2 \ldots, m_n$.
- Each configuration occurs at most once in every fundamental period.
- **The fundamental period is the number of occurring configurations.**
- Special case $m_1, m_2 \ldots, m_n$ pairwise coprime:
  The fundamental period is the product $m_1 \cdots m_n$, every configuration occurs.

CRT for rotations

We have thus proven:

Chinese Remainder Theorem for rotations

Let $m_1, \cdots, m_n$ be pairwise coprime positive integers. For each integer m in this set, consider a circle with one marked point on it, which at every time unit rotates of $\frac{1}{m}$-th of the circle clockwise. The fundamental period for this phenomenon is $m_1 \cdots m_n$, and every configuration for the n-tuple of marked points occurs exactly once in the fundamental period.

Remark: Even if the integer parameters are not coprime, the configuration of the marked points determines the time inside a fundamental period because it occurs at most once.

CRT for cyclic periodic phenomena

Consider a cyclic periodic phenomenon, where finitely many distinct situations repeat cyclically. This can be seen as a rotation with as many steps as the fundamental period, so we have proven:

Chinese Remainder Theorem for cyclic periodic phenomena

If $m_1, \cdots, m_n$ are pairwise coprime positive integers, then a cyclic periodic phenomenon with fundamental period $m_1 \cdots m_n$ amounts to the collection of simultaneous cyclic periodic phenomena whose fundamental periods are m_1 to m_n.

Corollary: A cyclic periodic phenomenon with fundamental period m can be described by cyclic periodic phenomena whose fundamental periods are the prime powers appearing in the factorization of m.

CRT for lists of remainders

Chinese Remainder Theorem for lists of remainders

Let $m_1, \cdots, m_n$ be pairwise coprime positive integers. For every integer consider the n-tuple of its remainders after division by m_1 to m_n. We then have:

- Two integers produce the same n-tuple if and only if they leave the same remainder after division by the product $m_1 \cdots m_n$.
- All n-tuples consisting of remainders after division by m_1 to m_n are produced.

Proof: The result can be deduced from the CRT for rotations. A list of remainders can be identified (in an obvious way) with a configuration of marked points. □

The usual CRT

Chinese Remainder Theorem

Let $m_1, \cdots, m_n$ be pairwise coprime positive integers. An integer in the range from 0 to $m_1 \cdots m_n - 1$ is uniquely determined by the n-tuple of its remainders after division by m_1 to m_n.

Proof: This is a reformulation of the previous result. □

Generalisation of the usual CRT

Generalized Chinese Remainder Theorem

Let $m_1, \cdots, m_n$ be positive integers. An integer in the range from 0 to $\mathrm{lcm}(m_1, \ldots m_n) - 1$ is uniquely determined by the n-tuple of its remainders after division by m_1 to m_n.

Proof: The result can be deduced from the corresponding result about simultaneous step-wise rotations. □

Remark that an impossible n-tuple of remainders corresponds to an impossible configuration for the n-tuple of marked points.

Thank you
for your attention!